

# Twisted Exponential Sums

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## Abstract

Let  $k$  be a finite field of characteristic  $p$ ,  $l$  a prime number distinct to  $p$ ,  $\psi : k \rightarrow \overline{\mathbf{Q}}_l^*$  a nontrivial additive character, and  $\chi : k^{*n} \rightarrow \overline{\mathbf{Q}}_l^*$  a character on  $k^{*n}$ . Then  $\psi$  defines an Artin-Schreier sheaf  $\mathcal{L}_\psi$  on the affine line  $\mathbf{A}_k^1$ , and  $\chi$  defines a Kummer sheaf  $\mathcal{K}_\chi$  on the  $n$ -dimensional torus  $\mathbf{T}_k^n$ . Let  $f \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  be a Laurent polynomial. It defines a  $k$ -morphism  $f : \mathbf{T}_k^n \rightarrow \mathbf{A}_k^1$ . In this paper, we calculate the dimensions and weights of  $H_c^i(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$  under some non-degeneracy conditions on  $f$ . Our results can be used to estimate sums of the form

$$\sum_{x_1, \dots, x_n \in k^*} \chi_1(f_1(x_1, \dots, x_n)) \cdots \chi_m(f_m(x_1, \dots, x_n)) \psi(f(x_1, \dots, x_n)),$$

where  $\chi_1, \dots, \chi_m : k^* \rightarrow \mathbf{C}^*$  are multiplicative characters,  $\psi : k \rightarrow \mathbf{C}^*$  is a nontrivial additive character, and  $f_1, \dots, f_m, f$  are Laurent polynomials.

**Key words:** Toric scheme, perverse sheaf, weight.

**Mathematics Subject Classification:** 14G15, 14F20, 11L40.

## 0. Introduction

Let  $k$  be a finite field with  $q$  elements of characteristic  $p$ , let  $\chi_1, \dots, \chi_m : k^* \rightarrow \mathbf{C}^*$  be nontrivial multiplicative characters, let  $\psi : k \rightarrow \mathbf{C}^*$  be a nontrivial additive character, and let

$$f_1(X_1, \dots, X_n), \dots, f_m(X_1, \dots, X_n), f(X_1, \dots, X_n) \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$$

be Laurent polynomials. We make the convention that  $\chi_i(0) = 0$  ( $i = 1, \dots, m$ ). In number theory, we are often lead to study the sum

$$S_1 = \sum_{x_1, \dots, x_n \in k^*} \chi_1(f_1(x_1, \dots, x_n)) \cdots \chi_m(f_m(x_1, \dots, x_n)) \psi(f(x_1, \dots, x_n)).$$

For this purpose, let's consider another sum

$$S_2 = \sum_{x_1, \dots, x_{n+m} \in k^*} \chi_1^{-1}(x_{n+1}) \cdots \chi_m^{-1}(x_{n+m}) \psi(f(x_1, \dots, x_n) + x_{n+1}f_1(x_1, \dots, x_n) + \cdots + x_{n+m}f_m(x_1, \dots, x_n)).$$

We have

$$\begin{aligned} S_2 &= \sum_{x_1, \dots, x_n \in k^*} \sum_{x_{n+1}, \dots, x_{n+m} \in k^*} (\chi_1^{-1}(x_{n+1}) \psi(x_{n+1}f_1(x_1, \dots, x_n))) \cdots (\chi_m^{-1}(x_{n+m}) \psi(x_{n+m}f_m(x_1, \dots, x_n))) \\ &\quad \psi(f(x_1, \dots, x_n)) \\ &= \sum_{x_1, \dots, x_n \in k^*} \left( \sum_{x_{n+1} \in k^*} \chi_1^{-1}(x_{n+1}) \psi(x_{n+1}f_1(x_1, \dots, x_n)) \right) \cdots \left( \sum_{x_{n+m} \in k^*} \chi_m^{-1}(x_{n+m}) \psi(x_{n+m}f_m(x_1, \dots, x_n)) \right) \\ &\quad \psi(f(x_1, \dots, x_n)). \end{aligned}$$

For  $i = 1, \dots, m$  and  $x_1, \dots, x_n \in k^*$ , if  $f_i(x_1, \dots, x_n) = 0$ , we have

$$\sum_{x_{n+i} \in k^*} \chi_i^{-1}(x_{n+i}) \psi(x_{n+i}f_i(x_1, \dots, x_n)) = 0;$$

if  $f_i(x_1, \dots, x_n) \neq 0$ , we have

$$\begin{aligned} \sum_{x_{n+i} \in k^*} \chi_i^{-1}(x_{n+i}) \psi(x_{n+i}f_i(x_1, \dots, x_n)) &= \sum_{x \in k^*} \chi_i^{-1} \left( \frac{x}{f_i(x_1, \dots, x_n)} \right) \psi(x) \\ &= \chi_i(f_i(x_1, \dots, x_n)) G(\chi_i, \psi), \end{aligned}$$

where

$$G(\chi_i, \psi) = \sum_{x \in k^*} \chi_i^{-1}(x) \psi(x)$$

is the Gauss sum. So in any case, we have

$$\sum_{x_{n+i} \in k^*} \chi_i^{-1}(x_{n+i}) \psi(x_{n+i}f_i(x_1, \dots, x_n)) = \chi_i(f_i(x_1, \dots, x_n)) G(\chi_i, \psi).$$

Hence

$$\begin{aligned} S_2 &= \sum_{x_1, \dots, x_n \in k^*} \chi_1(f_1(x_1, \dots, x_n)) G(\chi_1, \psi) \cdots \chi_m(f_m(x_1, \dots, x_n)) G(\chi_m, \psi) \psi(f(x_1, \dots, x_n)) \\ &= G(\chi_1, \psi) \cdots G(\chi_m, \psi) S_1. \end{aligned}$$

As the Gauss sums are well-understood, the study of  $S_1$  is reduced to the study of  $S_2$ .

In this paper, we use  $l$ -adic cohomology theory to study sums of the form

$$\sum_{x_i \in k^*} \chi_1(x_1) \cdots \chi_n(x_n) \psi(f(x_1, \dots, x_n)),$$

where  $\chi_1, \dots, \chi_n$  are multiplicative characters (nontrivial or trivial). Note that  $S_2$  is of this form.

Our results complete those in [DL], where the case of trivial  $\chi_i$  is treated. We follow the approach initiated by Denef and Loeser.

We first associate geometric objects to the above data. The Kummer covering

$$[q-1] : \mathbf{T}_k^n \rightarrow \mathbf{T}_k^n, \quad x \mapsto x^{q-1}$$

on the torus  $\mathbf{T}_k^n = \text{Spec } k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  defines a  $\mathbf{T}_k^n(k)$ -torsor

$$1 \rightarrow \mathbf{T}_k^n(k) \rightarrow \mathbf{T}_k^n \xrightarrow{[q-1]} \mathbf{T}_k^n \rightarrow 1,$$

where  $\mathbf{T}_k^n(k) = \text{Hom}_k(\text{Spec } k, \mathbf{T}_k^n)$  is the group of  $k$ -rational points in  $\mathbf{T}_k^n$ . Fix a prime number  $l$  distinct to  $p$ . Let  $\chi : \mathbf{T}_k^n(k) = k^{*n} \rightarrow \overline{\mathbf{Q}}_l^*$  be a character. Pushing-forward the above torsor by  $\chi^{-1}$ , we get a lisse  $\overline{\mathbf{Q}}_l$ -sheaf  $\mathcal{K}_\chi$  on  $\mathbf{T}_k^n$  of rank 1. We call  $\mathcal{K}_\chi$  the *Kummer sheaf* associated to  $\chi$ . For any rational point  $x$  in  $\mathbf{T}_k^n(k') = \text{Hom}_k(\text{Spec } k', \mathbf{T}_k^n)$  with value in a finite extension  $k'$  of  $k$ , we have

$$\text{Tr}(F_x, (\mathcal{K}_\chi)_{\bar{x}}) = \chi(\text{Norm}_{k'/k}(x)),$$

where  $F_x$  is the geometric Frobenius element at  $x$ .

The Artin-Schreier covering

$$\mathcal{P} : \mathbf{A}_k^1 \rightarrow \mathbf{A}_k^1, \quad x \mapsto x^q - x$$

defines an  $\mathbf{A}_k^1(k)$ -torsor

$$0 \rightarrow \mathbf{A}_k^1(k) \rightarrow \mathbf{A}_k^1 \xrightarrow{\mathcal{P}} \mathbf{A}_k^1 \rightarrow 0,$$

where  $\mathbf{A}_k^1(k) = \text{Hom}_k(\text{Spec } k, \mathbf{A}_k^1)$  is the group of  $k$ -rational points in  $\mathbf{A}_k^1$ . Let  $\psi : \mathbf{A}_k^1(k) = k \rightarrow \overline{\mathbf{Q}}_l^*$  be an additive character. Pushing-forward this torsor by  $\psi^{-1}$ , we get a lisse  $\mathbf{Q}_l$ -sheaf  $\mathcal{L}_\psi$  of rank 1 on  $\mathbf{A}_k^1$ , which we call the *Artin-Schreier sheaf*. For any rational point  $x$  in  $\mathbf{A}_k^1(k') = \text{Hom}_k(\text{Spec } k', \mathbf{A}_k^1)$  with value in a finite extension  $k'$  of  $k$ , we have

$$\text{Tr}(F_x, (\mathcal{L}_\psi)_{\bar{x}}) = \psi(\text{Tr}_{k'/k}(x)),$$

where  $F_x$  is the geometric Frobenius element at  $x$ .

Let

$$f = \sum_{i \in \mathbf{Z}^n} a_i X^i \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$$

be a Laurent polynomial. The *Newton polyhedron*  $\Delta_\infty(f)$  of  $f$  at  $\infty$  is the convex hull in  $\mathbf{R}^n$  of the set  $\{i \in \mathbf{Z}^n \mid a_i \neq 0\} \cup \{0\}$ . We say  $f$  is *non-degenerate* with respect to  $\Delta_\infty(f)$  if for any face  $\tau$  of  $\Delta_\infty(f)$  not containing 0, the locus of

$$\frac{\partial f_\tau}{\partial X_1} = \dots = \frac{\partial f_\tau}{\partial X_n} = 0$$

in  $\mathbf{T}_k^n$  is empty, where

$$f_\tau = \sum_{i \in \tau} a_i X^i.$$

This is equivalent to saying that the morphism

$$f_\tau : \mathbf{T}_k^n = \text{Spec } A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] \rightarrow \mathbf{A}_k^1 = \text{Spec } k[T]$$

defined by the  $k$ -algebra homomorphism

$$k[T] \rightarrow k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}], \quad T \mapsto f_\tau$$

is smooth.

The first main result of this paper is the following theorem.

**Theorem 0.1.** Let  $f : \mathbf{T}_k^n \rightarrow \mathbf{A}_k^1$  be a  $k$ -morphism defined by a Laurent polynomial  $f \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  that is non-degenerate with respect to  $\Delta_\infty(f)$  and let  $\mathcal{K}_\chi$  be a Kummer sheaf on  $\mathbf{T}_k^n$ . Suppose  $\dim(\Delta_\infty(f)) = n$ . Then

- (i)  $H_c^i(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi) = 0$  for  $i \neq n$ .
- (ii)  $\dim(H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)) = n! \text{vol}(\Delta_\infty(f))$ .

(iii) If 0 is an interior point of  $\Delta_\infty(f)$ , then  $H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$  is pure of weight  $n$ .

Here the conclusion of (iii) means that for any eigenvalue  $\lambda$  of the geometric Frobenius element  $F$  in  $\text{Gal}(\bar{k}/k)$  acting on  $H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$ ,  $\lambda$  is an algebraic number, and all the galois conjugates of  $\lambda$  have archimedean absolute value  $q^{\frac{n}{2}}$ .

Note that we have

$$[q-1]_*[q-1]^* f^* \mathcal{L}_\psi \cong \bigoplus_{\chi} (\mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi),$$

where  $[q-1] : \mathbf{T}_k^n \rightarrow \mathbf{T}_k^n$  is the Kummer covering, and in the direct sum on the right-hand side,  $\chi$  goes over the set of all characters  $\chi : \mathbf{T}_k^n(k) \rightarrow \bar{\mathbf{Q}}_l^\times$ . The composition  $f \circ [q-1]$  is defined by the Laurent polynomial

$$f'(X_1, \dots, X_n) = f(X_1^{q-1}, \dots, X_n^{q-1}).$$

Note that  $f'$  is also non-degenerate with respect to its Newton polyhedron at  $\infty$ . We have

$$\begin{aligned} H_c^i(\mathbf{T}_k^n, f'^* \mathcal{L}_\psi) &\cong H_c^i(\mathbf{T}_k^n, [q-1]_*[q-1]^* f^* \mathcal{L}_\psi) \\ &\cong \bigoplus_{\chi} H_c^i(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi). \end{aligned}$$

So  $H_c^i(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$  are direct factors of  $H_c^i(\mathbf{T}_k^n, f'^* \mathcal{L}_\psi)$ . Hence Theorem 0.1 (i) and (iii) follow directly from the main theorem 1.3 in [DL] applied to  $f'$ . Using [I1] 2.1, one can show

$$\chi_c(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi) = \chi_c(\mathbf{T}_k^n, f^* \mathcal{L}_\psi),$$

where  $\chi_c$  denotes the Euler characteristic for the cohomology with compact support. Hence Theorem 0.1 (ii) can also be deduced from [DL] 1.3.

In this paper, we give a proof of Theorem 0.1 independent of the main theorem of [DL]. On the other hand, our second main Theorem 0.4 on the weights of  $H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$  doesn't seem to follow from the corresponding theorem in [DL].

**Corollary 0.2.** Let  $f \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  be a Laurent polynomial that is non-degenerate with respect to  $\Delta_\infty(f)$  and suppose  $\dim(\Delta_\infty(f)) = n$ . Then for any multiplicative characters  $\chi_1, \dots, \chi_n : k^* \rightarrow \mathbf{C}^*$  and any nontrivial additive character  $\psi : k \rightarrow \mathbf{C}^*$ , we have

$$\left| \sum_{x_i \in k^*} \chi_1(x_1) \cdots \chi_n(x_n) \psi(f(x_1, \dots, x_n)) \right| \leq n! \text{vol}(\Delta_\infty(f)) q^{\frac{n}{2}}.$$

**Proof.** Note that the values of  $\chi_i$  and  $\psi$  are algebraic integers. In particular, we may consider them to have values in  $\overline{\mathbf{Q}}_l$ . Let  $\chi : k^{*n} \rightarrow \overline{\mathbf{Q}}_l^*$  be the character defined by

$$\chi(x) = \chi_1(x_1) \cdots \chi_n(x_n)$$

for any  $x = (x_1, \dots, x_n) \in k^{*n}$ . We then have

$$\sum_{x_i \in k^*} \chi_1(x_1) \cdots \chi_n(x_n) \psi(f(x_1, \dots, x_n)) = \sum_{x \in \mathbf{T}_k^n(k)} \mathrm{Tr}(F_x, (\mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)_{\bar{x}}).$$

By the Grothendieck trace formula ([SGA 4 $\frac{1}{2}$ ], [Rapoport] Théorème 3.2), we have

$$\sum_{x \in \mathbf{T}_k^n(k)} \mathrm{Tr}(F_x, (\mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)_{\bar{x}}) = \sum_{i=0}^{2n} (-1)^i \mathrm{Tr}(F, H_c^i(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)).$$

By [D] 3.3.1, for any eigenvalue  $\lambda$  of  $F$  acting on  $H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$ ,  $\lambda$  is an algebraic number and all its galois conjugates have archimedean absolute value  $\leq q^{\frac{n}{2}}$ . Combined with Theorem 0.1 (i) and (ii), we get

$$|\sum_{i=0}^{2n} (-1)^i \mathrm{Tr}(F, H_c^i(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi))| \leq n! \mathrm{vol}(\Delta_\infty(f)) q^{\frac{n}{2}}.$$

So we have

$$|\sum_{x_i \in k^*} \chi_1(x_1) \cdots \chi_n(x_n) \psi(f(x_1, \dots, x_n))| \leq n! \mathrm{vol}(\Delta_\infty(f)) q^{\frac{n}{2}}.$$

Combining Corollary 0.2 with the discussion at the beginning, and using the fact that Gauss sums have absolute value  $q^{\frac{1}{2}}$ , we get the following.

**Corollary 0.3.** Let  $f, f_1, \dots, f_m \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  be Laurent polynomials,  $\chi_1, \dots, \chi_m : k^* \rightarrow \mathbf{C}^*$  nontrivial multiplicative characters, and  $\psi : k \rightarrow \mathbf{C}^*$  a nontrivial additive character. Suppose the Laurent polynomial

$$F(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) = f(x_1, \dots, x_n) + x_{n+1} f_1(x_1, \dots, x_n) + \dots + x_{n+m} f_m(x_1, \dots, x_n)$$

is non-degenerate with respect to  $\Delta_\infty(F)$ , and  $\dim(\Delta_\infty(F)) = m + n$ . Then we have

$$|\sum_{x_i \in k^*} \chi_1(f_1(x_1, \dots, x_n)) \cdots \chi_m(f_m(x_1, \dots, x_n)) \psi(f(x_1, \dots, x_n))| \leq (n+m)! \mathrm{vol}(\Delta_\infty(F)) q^{n/2}.$$

Under the assumption of Theorem 0.1, let

$$E(\mathbf{T}_k^n, f, \chi) = \sum_{w \in \mathbf{Z}} e_w T^w,$$

where  $e_w$  is the number of eigenvalues counted with multiplicities of the geometric Frobenius element  $F$  in  $\text{Gal}(\bar{k}/k)$  acting on  $H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$  with weight  $w$ . Our next goal is to determine  $E(\mathbf{T}_k^n, f, \chi)$ .

For any convex polyhedral cone  $\delta$  in  $\mathbf{R}^n$  with  $0$  being a face, define the convex polytope  $\text{poly}(\delta)$  to be the intersection of  $\delta$  with a hyperplane in  $\mathbf{R}^n$  which does not contain  $0$  and intersects each one dimensional face of  $\delta$ . Note that  $\text{poly}(\delta)$  is defined only up to combinatorial equivalence. For any convex polytope  $\Delta$  in  $\mathbf{R}^n$  and any face  $\tau$  of  $\Delta$ , define  $\text{cone}_\Delta(\tau)$  to be the cone generated by  $u' - u$  ( $u' \in \Delta$ ,  $u \in \tau$ ), and define  $\text{cone}_\Delta^\circ(\tau)$  to be the image of  $\text{cone}_\Delta(\tau)$  in  $\mathbf{R}^n / \text{span}(\tau - \tau)$ . Note that  $0$  is a face of  $\text{cone}_\Delta^\circ(\tau)$ . We define polynomials  $\alpha(\delta)$  and  $\beta(\Delta)$  in one variable  $T$  inductively by the following formulas:

$$\begin{aligned} \alpha(\{0\}) &= 1, \\ \beta(\Delta) &= (T^2 - 1)^{\dim(\Delta)} + \sum_{\tau \text{ face of } \Delta, \tau \neq \Delta} (T^2 - 1)^{\dim(\tau)} \alpha(\text{cone}_\Delta^\circ(\tau)), \\ \alpha(\delta) &= \text{trunc}_{\leq \dim(\delta)-1}((1 - T^2)\beta(\text{poly}(\delta))), \end{aligned}$$

where  $\text{trunc}_{\leq d}(\cdot)$  denotes taking the degree  $\leq d$  part of a polynomial. These polynomials are first introduced by Stanley [S]. Note that  $\alpha(\delta)$  and  $\beta(\Delta)$  only involve even powers of  $T$ , and they depend only on the combinatorial types of  $\delta$  and  $\Delta$ . If  $\delta$  is a simplicial cone, that is, if  $\delta$  is generated by linearly independent vectors, then using induction on  $\dim(\delta)$ , one can verify  $\alpha(\delta) = 1$ .

Let  $\chi : \mathbf{T}_k^n(k) \rightarrow \overline{\mathbf{Q}}_l^*$  be a character. For a rational convex polytope  $\Delta$  in  $\mathbf{R}^n$  of dimension  $n$ , let  $T$  be the set of faces  $\tau$  of  $\Delta$  so that  $\tau \neq \Delta$ ,  $0 \in \tau$ , and  $\mathcal{K}_\chi \cong p_\tau^* \mathcal{K}_\tau$  for a Kummer sheaf  $\mathcal{K}_\tau$  on the torus  $\mathbf{T}_\tau = \text{Spec } k[\mathbf{Z}^n \cap \text{span}(\tau - \tau)]$ , where

$$p_\tau : \mathbf{T}_k^n = \text{Spec } k[\mathbf{Z}^n] \rightarrow \mathbf{T}_\tau = \text{Spec } k[\mathbf{Z}^n \cap \text{span}(\tau - \tau)]$$

is the morphism defined by the canonical homomorphism

$$k[\mathbf{Z}^n \cap \text{span}(\tau - \tau)] \hookrightarrow k[\mathbf{Z}^n].$$

Define

$$e(\Delta, \chi) = n! \text{vol}(\Delta) + \sum_{\tau \in T} (-1)^{n-\dim(\tau)} (\dim(\tau))! \text{vol}(\tau) \alpha(\text{cone}_\Delta^\circ(\tau))(1)$$

and define a polynomial  $E(\Delta, \chi)$  inductively by

$$E(\Delta, \chi) = e(\Delta, \chi) T^n - \sum_{\tau \in T} (-1)^{n-\dim(\tau)} E(\tau, \chi_\tau) \alpha(\text{cone}_\Delta^\circ(\tau)).$$

Our second main result is the following.

**Theorem 0.4.** Let  $f : \mathbf{T}_k^n \rightarrow \mathbf{A}_k^1$  be a  $k$ -morphism defined by a Laurent polynomial  $f \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  that is non-degenerate with respect to  $\Delta_\infty(f)$  and let  $\mathcal{K}_\chi$  be a Kummer sheaf on  $\mathbf{T}_k^n$ . Suppose  $\dim(\Delta_\infty(f)) = n$ . Let

$$E(\mathbf{T}_k^n, f, \chi) = \sum_{w \in \mathbf{Z}} e_w T^w,$$

where  $e_w$  is the number of eigenvalues counted with multiplicities of the geometric Frobenius element  $F$  in  $\text{Gal}(\bar{k}/k)$  acting on  $H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$  with weight  $w$ . Then  $E(\mathbf{T}_k^n, f, \chi)$  is a polynomial of degree  $\leq n$ , and

$$\begin{aligned} E(\mathbf{T}_k^n, f, \chi) &= E(\Delta_\infty(f), \chi), \\ e_n &= e(\Delta_\infty(f), \chi). \end{aligned}$$

**Corollary 0.5.** Let  $f : \mathbf{T}_k^n \rightarrow \mathbf{A}_k^1$  be a  $k$ -morphism defined by a Laurent polynomial  $f \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  that is non-degenerate with respect to  $\Delta_\infty(f)$  and let  $\mathcal{K}_\chi$  be a Kummer sheaf on  $\mathbf{T}_k^n$ . Suppose  $\dim(\Delta_\infty(f)) = n$  and suppose for any face  $\tau$  of  $\Delta_\infty(f)$  of codimension one containing 0,  $\mathcal{K}_\chi$  is not the inverse image of any Kummer sheaf on  $\mathbf{T}_\tau = \text{Spec } k[\mathbf{Z}^n \cap \text{span}(\tau - \tau)]$  under the morphism

$$p_\tau : \mathbf{T}_k^n = \text{Spec } k[\mathbf{Z}^n] \rightarrow \mathbf{T}_\tau = \text{Spec } k[\mathbf{Z}^n \cap \text{span}(\tau - \tau)].$$

Then  $H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$  is pure of weight  $n$ .

**Remark 0.6.** Note that this corollary together with Theorem 0.1 is Theorem 4.2 in [AS], except that Adolphson and Sperber prove the theorem for almost all  $p$ .



**Proof of Corollary 0.5.** Under our assumption, the set  $T$  of faces  $\tau$  of  $\Delta_\infty(f)$  so that  $\tau \neq \Delta_\infty(f)$ ,  $0 \in \tau$ , and  $\mathcal{K}_\chi \cong p_\tau^* \mathcal{K}_\tau$  for a Kummer sheaf  $\mathcal{K}_\tau$  on the torus  $\mathbf{T}_\tau = \text{Spec } k[\mathbf{Z}^n \cap \text{span}(\tau - \tau)]$  is empty. Therefore

$$E(\Delta_\infty(f), \chi) = e(\Delta_\infty(f), \chi) T^n.$$

By Theorem 0.4, this implies

$$E(\mathbf{T}_k^n, f, \chi) = e(\Delta_\infty(f), \chi) T^n,$$

and hence  $H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$  is pure of weight  $n$ .

For a rational convex polytope  $\Delta$  in  $\mathbf{R}^n$  with dimension  $n$ , recall that  $T$  is the set of faces  $\tau$  of  $\Delta$  so that  $\tau \neq \Delta$ ,  $0 \in \tau$ , and  $\mathcal{K}_\chi \cong p_\tau^* \mathcal{K}_\tau$  for a Kummer sheaf  $\mathcal{K}_\tau$  on the torus  $\mathbf{T}_\tau = \text{Spec } k[\mathbf{Z}^n \cap \text{span}(\tau - \tau)]$ . Define

$$V_i(\Delta, \chi) = \sum_{\tau \in T, \dim(\tau)=i} \text{vol}(\tau)$$

for  $0 \leq i \leq n-1$  and define

$$V_n(\Delta, \chi) = \text{vol}(\Delta).$$

Let  $\tau_0$  be the smallest face of  $\Delta$  containing 0. We say  $\Delta$  is *simplicial at the origin* if there are exactly  $n - \dim(\tau_0)$  faces of  $\Delta$  that have dimension  $n-1$  and contain  $\tau_0$ . If  $\Delta$  is simplicial at the origin and  $\tau$  is a face containing 0, then the number of faces of  $\Delta$  that have dimension  $k$  and contain  $\tau$  is  $\binom{n - \dim(\tau)}{n - k}$ .

The following corollary is Theorem 4.8 in [AS].

**Corollary 0.7.** Let  $f : \mathbf{T}_k^n \rightarrow \mathbf{A}_k^1$  be a  $k$ -morphism defined by a Laurent polynomial  $f \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  that is non-degenerate with respect to  $\Delta_\infty(f)$  and let  $\mathcal{K}_\chi$  be a Kummer sheaf on  $\mathbf{T}_k^n$ . Suppose  $\dim(\Delta_\infty(f)) = n$  and suppose  $\Delta_\infty(f)$  is simplicial at the origin. Let  $e_w$  be the number of eigenvalues counted with multiplicities of the geometric Frobenius element  $F$  in  $\text{Gal}(\bar{k}/k)$  acting on  $H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$  with weight  $w$ . Then we have

$$e_w = \sum_{i=0}^w (-1)^{w-i} i! \binom{n-i}{n-w} V_i(\Delta_\infty(f), \chi).$$

**Proof.** The hypothesis that  $\Delta_\infty(f)$  is simplicial at the origin implies that

$$\alpha(\text{cone}_{\Delta_\infty(f)}^\circ(\tau)) = 1$$

for any face  $\tau$  of  $\Delta_\infty(f)$  containing 0. By Theorem 0.4, we have

$$\begin{aligned}
e_n &= e(\Delta_\infty(f), \chi) \\
&= n! \text{vol}(\Delta_\infty(f)) + \sum_{\tau \in T} (-1)^{n-\dim(\tau)} (\dim(\tau))! \text{vol}(\tau) \\
&= n! V_n(\Delta_\infty(f), \chi) + \sum_{i=0}^{n-1} (-1)^{n-i} i! V_i(\Delta_\infty(f), \chi) \\
&= \sum_{i=0}^n (-1)^{n-i} i! V_i(\Delta_\infty(f), \chi).
\end{aligned}$$

This proves our assertion for  $w = n$ . We use induction on  $n$ . Under our assumption, we have

$$E(\Delta_\infty(f), \chi) = e(\Delta_\infty(f), \chi) T^n - \sum_{\tau \in T} (-1)^{n-\dim(\tau)} E(\tau, \chi_\tau).$$

For  $w \leq n-1$ , taking the coefficients of  $T^w$  on both sides of the above equality and applying Theorem 0.4 and the induction hypothesis to the pairs  $(\tau, \chi_\tau)$  ( $\tau \in T$ ), we get

$$e_w = - \sum_{\tau \in T, w \leq \dim(\tau) \leq n-1} (-1)^{n-\dim(\tau)} \sum_{i=0}^w (-1)^{w-i} i! \binom{\dim(\tau) - i}{\dim(\tau) - w} \sum_{\tau' \prec \tau, \dim(\tau')=i, \tau' \in T} \text{vol}(\tau').$$

So we have

$$e_w = \sum_{i=0}^w (-1)^{w-i+n+1} i! \sum_{\tau' \in T, \dim(\tau')=i} \sum_{\tau' \prec \tau, w \leq \dim(\tau) \leq n-1} (-1)^{\dim(\tau)} \binom{\dim(\tau) - i}{\dim(\tau) - w} \text{vol}(\tau').$$

By our assumption, for each  $\tau'$  containing 0 of dimension  $i$ , the number of faces of  $\Delta_\infty(f)$  that have dimension  $k$  and contain  $\tau'$  is  $\binom{n-i}{n-k}$ . So we have

$$e_w = \sum_{i=0}^w (-1)^{w-i+n+1} i! \sum_{\tau' \in T, \dim(\tau')=i} \sum_{k=w}^{n-1} (-1)^k \binom{k-i}{k-w} \binom{n-i}{n-k} \text{vol}(\tau').$$

We have

$$\begin{aligned}
& \sum_{k=w}^{n-1} (-1)^k \binom{k-i}{k-w} \binom{n-i}{n-k} \\
&= \sum_{k=w}^{n-1} (-1)^k \binom{n-w}{k-w} \binom{n-i}{n-w} \\
&= (-1)^w \binom{n-i}{n-w} \sum_{j=0}^{n-1-w} (-1)^j \binom{n-w}{j} \\
&= (-1)^{n-1} \binom{n-i}{n-w}.
\end{aligned}$$

So we have

$$\begin{aligned}
e_w &= \sum_{i=0}^w (-1)^{w-i+n+1} i! \sum_{\tau' \in T, \dim(\tau')=i} (-1)^{n-1} \binom{n-i}{n-w} \text{vol}(\tau'). \\
&= \sum_{i=0}^w (-1)^{w-i} i! \binom{n-i}{n-w} V_i(\Delta_\infty(f), \chi).
\end{aligned}$$

This proves our assertion.

The paper is organized as follows. In §1, we summarize basic results on toric schemes. These results can be found in [F]. This section is mainly for the purpose of fixing notations. A toric scheme contains an open dense torus. In §2, we study extensions of Kummer sheaves on tori to toric schemes. In §3, we study the compactifications by toric schemes of a morphism on a torus defined by a Laurent polynomial. In §4, we prove our main results.

**Acknowledgement.** The research is supported by the NSFC (10525107).

## 1. Toric Schemes

In this paper, a *lattice*  $N$  is a free abelian group of finite rank. Let  $M = \text{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$  be the dual lattice of  $N$ , let  $V = N \otimes_{\mathbf{Z}} \mathbf{R}$  be the real vector space generated by  $N$ , and let  $V^* = \text{Hom}_{\mathbf{R}}(V, \mathbf{R})$  be the dual vector space of  $V$ . We have a canonical identification  $M \otimes_{\mathbf{Z}} \mathbf{R} \cong V^*$ . A *convex polyhedral cone*  $\sigma$  in  $V$  is a subset of the form

$$\sigma = \{r_1 v_1 + \cdots + r_k v_k \mid r_i \geq 0\},$$

where  $v_1, \dots, v_k$  is a finite family of elements in  $V$ , which is called a family of *generators* of  $\sigma$ . We say  $\sigma$  is *rational* with respect to the lattice  $N$  if  $v_1, \dots, v_k$  can be chosen to lie in  $N$ . Define the *dual*  $\check{\sigma}$  of  $\sigma$  to be

$$\check{\sigma} = \{u \in V^* \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

If  $\sigma$  is a rational convex polyhedral cone in  $V$  with respect to  $N$ , then  $\check{\sigma}$  is a rational convex polyhedral cone in  $V^*$  with respect to  $M$  ([F] 1.2 (9)). Under this condition, the semigroup  $M \cap \check{\sigma}$  is finitely generated ([F] 1.2, Proposition 1). Let  $A$  be a commutative ring. Then the ring  $A[M \cap \check{\sigma}]$

is a finitely generated  $A$ -algebra. For any  $u \in M \cap \check{\sigma}$ , denote the corresponding element in  $A[M \cap \check{\sigma}]$  by  $\chi^u$ . We have

$$\chi^{u_1+u_2} = \chi^{u_1} \chi^{u_2}$$

in  $A[M \cap \check{\sigma}]$  for any  $u_1, u_2 \in M \cap \check{\sigma}$ . A *face*  $\tau$  of  $\sigma$  is a subset of the form

$$\tau = \sigma \cap u^\perp = \{v \in \sigma \mid \langle u, v \rangle = 0\}$$

for some  $u \in \check{\sigma}$ . It is also a convex polyhedral cone ([F] 1.2 (2)). We use the notation  $\tau \prec \sigma$  or  $\sigma \succ \tau$  to denote  $\tau$  being a face of  $\sigma$ . When  $\sigma$  is rational, so is  $\tau$ , and we may then choose  $u \in M \cap \check{\sigma}$ . We then have

$$M \cap \check{\tau} = M \cap \check{\sigma} + \mathbf{Z}_{\geq 0}(-u)$$

by [F] 1.2, Proposition 2, and hence

$$A[M \cap \check{\tau}] = A[M \cap \check{\sigma}]_{\chi^u}.$$

Define

$$U_\sigma = \text{Spec } A[M \cap \check{\sigma}].$$

Then the canonical homomorphism

$$A[M \cap \check{\sigma}] \hookrightarrow A[M \cap \check{\tau}]$$

defines an open immersion

$$U_\tau \hookrightarrow U_\sigma.$$

A *fan*  $\Sigma$  is a finite family of rational convex polyhedral cones in  $V$  satisfying the following properties:

- (a)  $0$  is face of each cone in  $\Sigma$ .
- (b) A face of a cone in  $\Sigma$  is also in  $\Sigma$ .
- (c) If  $\sigma, \sigma' \in \Sigma$ , then  $\sigma \cap \sigma'$  is a face of both  $\sigma$  and  $\sigma'$ .

For any  $\sigma, \sigma' \in \Sigma$ , by the discussion above,  $U_{\sigma \cap \sigma'}$  can be considered as an open subscheme of both  $U_\sigma$  and  $U_{\sigma'}$ . Gluing  $U_\sigma$  and  $U_{\sigma'}$  along  $U_{\sigma \cap \sigma'}$ , we get a scheme  $X_A(\Sigma)$  of finite type over  $A$ , which we call the *toric scheme* associated to the fan  $\Sigma$ . The scheme  $X_A(\Sigma)$  is separated over  $A$  ([F] 1.4, Lemma). If  $\bigcup_{\sigma \in \Sigma} \sigma = V$ , then it is proper over  $A$  ([F] 2.4, Proposition). A rational convex polyhedral cone  $\sigma$  is called *regular* if it can be generated by part of a basis of  $N$ . Under

this condition,  $U_\sigma$  is smooth over  $A$  ([F] 2.1). A fan  $\Sigma$  is called *regular* if all the cones in  $\Sigma$  are regular. Under this condition,  $X_A(\Sigma)$  is smooth over  $A$ .

Each  $U_\sigma$  ( $\sigma \in \Sigma$ ) can be regarded as an open subscheme of  $X_A(\Sigma)$ . Taking  $\sigma = 0$ , we see the torus

$$U_0 = \operatorname{Spec} A[M]$$

is an open subscheme of  $X_A(\Sigma)$ . One can show this torus is dense in  $X_A(\Sigma)$ . Let  $n$  be the rank of  $N$ . Then the torus  $\operatorname{Spec} A[M]$  has relative dimension  $n$ , and we denote it by  $\mathbf{T}_A^n$ . For any  $\sigma \in \Sigma$ , the  $A$ -algebra homomorphism

$$A[M \cap \check{\sigma}] \rightarrow A[M] \otimes_A A[M \cap \check{\sigma}], \chi^u \mapsto \chi^u \otimes \chi^u \ (u \in M \cap \check{\sigma})$$

defines an action

$$\mathbf{T}_A^n \times_A U_\sigma \rightarrow U_\sigma.$$

These actions for  $\sigma \in \Sigma$  can be glued together to give an action

$$\mathbf{T}_A^n \times_A X_A(\Sigma) \rightarrow X_A(\Sigma)$$

which extends the action of  $\mathbf{T}_A^n$  on itself.

For any  $\tau \in \Sigma$ , let

$$N_\tau = N \cap \tau + (-N \cap \tau) = N \cap \operatorname{span}(\tau)$$

be the group generated by  $N \cap \tau$ , and let  $N(\tau) = N/N_\tau$ . Then  $N(\tau)$  is torsion free, and hence a lattice. Let  $M(\tau)$  be its dual lattice. We have a canonical isomorphism

$$M(\tau) \cong M \cap \tau^\perp.$$

For each  $\sigma \in \Sigma$  with  $\tau \prec \sigma$ , let  $\bar{\sigma}$  be the image of  $\sigma$  in  $N(\tau) \otimes_{\mathbf{Z}} \mathbf{R} \cong V/\operatorname{span}(\tau)$ . Note that  $\sigma$  is completely determined by  $\bar{\sigma}$ , that is, for any  $\sigma, \sigma' \in \Sigma$  containing  $\tau$ , we have  $\sigma = \sigma'$  if and only if  $\bar{\sigma} = \bar{\sigma}'$ . The family

$$\operatorname{star}(\tau) = \{\bar{\sigma} | \sigma \in \Sigma, \tau \prec \sigma\}$$

is a fan in  $N(\tau) \otimes_{\mathbf{Z}} \mathbf{R}$ . Let

$$V(\tau) = X_A(\operatorname{star}(\tau)).$$

It is obtained by gluing

$$U_\sigma(\tau) = \text{Spec } A[M(\tau) \cap \check{\sigma}] = \text{Spec } A[M \cap \check{\sigma} \cap \tau^\perp]$$

together ( $\sigma \in \text{Star}(\tau)$ ). Let

$$O_\tau = U_\tau(\tau) = \text{Spec } A[M \cap \tau^\perp]$$

be the open dense torus in  $V(\tau)$ . Its relative dimension is  $\dim(V) - \dim(\tau)$ . For any  $\sigma \in \Sigma$  with  $\tau \prec \sigma$ , consider the map

$$\begin{aligned} M \cap \check{\sigma} &\rightarrow A[M \cap \check{\sigma} \cap \tau^\perp], \\ u &\mapsto \begin{cases} \chi^u & \text{if } u \in \check{\sigma} \cap \tau^\perp, \\ 0 & \text{if } u \notin \check{\sigma} \cap \tau^\perp. \end{cases} \end{aligned}$$

It is a semigroup homomorphism, where the semigroup law on  $A[M \cap \check{\sigma} \cap \tau^\perp]$  is multiplication. (To prove this, we use the fact that  $\check{\sigma} \cap \tau^\perp$  is a face of  $\check{\sigma}$  ([F] 1.2 (10)), and a sum of elements in a cone lies in a face if and only if each summand lies in the face.) It induces an  $A$ -algebra epimorphism

$$A[M \cap \check{\sigma}] \rightarrow A[M \cap \check{\sigma} \cap \tau^\perp]$$

and hence a closed immersion

$$U_\sigma(\tau) \rightarrow U_\sigma.$$

For any  $\sigma, \sigma' \in \Sigma$  with  $\tau \prec \sigma' \prec \sigma$ , the diagram

$$\begin{array}{ccc} U_{\sigma'}(\tau) & \rightarrow & U_{\sigma'} \\ \downarrow & & \downarrow \\ U_\sigma(\tau) & \rightarrow & U_\sigma \end{array}$$

commutes and is Cartesian. So we can glue these closed immersions together to get a closed immersion

$$V(\tau) \rightarrow \bigcup_{\tau \prec \sigma} U_\sigma.$$

One can show for those  $\sigma \in \Sigma$  not containing  $\tau$ , the image of  $V(\tau)$  in  $\bigcup_{\tau \prec \sigma} U_\sigma$  is disjoint from  $U_\sigma$ . So composing the above closed immersion with the open immersion  $\bigcup_{\tau \prec \sigma} U_\sigma \rightarrow X_A(\Sigma)$ , we get a closed immersion

$$V(\tau) \rightarrow X_A(\Sigma).$$

We regard the open dense torus  $O_\tau$  of  $V(\tau)$  as a subscheme of  $X_A(\Sigma)$  through this closed immersion. One can verify  $O_\tau$  ( $\tau \in \Sigma$ ) are disjoint in  $X_A(\Sigma)$ . Actually  $O_\tau$  are the orbits of the torus action

$\mathbf{T}_A^n$  on  $X_A(\Sigma)$ . Moreover, we have ([F] 3.1, Proposition)

$$\begin{aligned} U_\sigma &= \coprod_{\gamma \prec \sigma} O_\gamma, \\ V(\tau) &= \coprod_{\tau \prec \gamma} O_\gamma. \end{aligned}$$

Let  $\delta$  be a rational convex polyhedral cone of dimension  $\dim(V)$  in the dual space  $V^*$ . We have

$$\check{\delta} \cap (-\check{\delta}) = \delta^\perp = 0.$$

So by [F] 1.2 (10), 0 is a face of  $\check{\delta}$ , and hence the family  $\Sigma(\delta)$  of faces of  $\check{\delta}$  is a fan in  $V$ , and we have

$$X_A(\Sigma(\delta)) = U_{\check{\delta}} = \text{Spec } A[M \cap \delta].$$

By [F] 1.2 (10), the map  $\tau \mapsto \check{\delta} \cap \tau^\perp$  sets up a one-to-one correspondence between the family of faces of  $\delta$  and the family of faces of  $\check{\delta}$ . We claim

$$\check{\delta} \cap \tau^\perp = (\text{cone}_\delta(\tau))^\vee,$$

where  $\text{cone}_\delta(\tau)$  is the cone in  $V^*$  generated by  $u' - u$  ( $u' \in \delta$ ,  $u \in \tau$ ). Indeed, if  $v \in (\text{cone}_\delta(\tau))^\vee$ , then for any  $u' \in \delta$ ,  $u \in \tau$ , we have

$$\langle u', v \rangle \geq \langle u, v \rangle.$$

Taking  $u, u' \in \tau$ , we see  $\langle \cdot, v \rangle|_\tau$  is constant. As  $0 \in \tau$ , we have  $\langle \cdot, v \rangle|_\tau = 0$ , that is,  $v \in \tau^\perp$ . The above inequality then implies that  $\langle u', v \rangle \geq 0$  for all  $u' \in \delta$ , that is,  $v \in \check{\delta}$ . So we have  $v \in \check{\delta} \cap \tau^\perp$ . Hence  $(\text{cone}_\delta(\tau))^\vee \subset \check{\delta} \cap \tau^\perp$ . It is not hard to see  $\check{\delta} \cap \tau^\perp \subset (\text{cone}_\delta(\tau))^\vee$ . So  $\check{\delta} \cap \tau^\perp = (\text{cone}_\delta(\tau))^\vee$ . By [F] 1.2 (10), we have

$$\begin{aligned} \dim \left( (\text{cone}_\delta(\tau)) \cap (-\text{cone}_\delta(\tau)) \right) &= \dim(V) - \dim((\text{cone}_\delta(\tau))^\vee) \\ &= \dim(V) - \dim(\check{\delta} \cap \tau^\perp) \\ &= \dim(\tau). \end{aligned}$$

As

$$\text{span}(\tau) = \tau - \tau \subset (\text{cone}_\delta(\tau)) \cap (-\text{cone}_\delta(\tau))$$

and  $(\text{cone}_\delta(\tau)) \cap (-\text{cone}_\delta(\tau))$  is a linear space, we have

$$(\text{cone}_\delta(\tau)) \cap (-\text{cone}_\delta(\tau)) = \tau - \tau.$$

We summarize these results as follows.

**Proposition 1.1.** Let  $\delta$  be a rational convex polyhedral cone of dimension  $\dim(V)$  in  $V^*$ . For any face  $\tau$  of  $\delta$ , let  $\text{cone}_\delta(\tau)$  be the cone in  $V^*$  generated by  $u' - u$  ( $u' \in \delta, u \in \tau$ ). We have

$$(\text{cone}_\delta(\tau))^\vee = \check{\delta} \cap \tau^\perp.$$

The family

$$\Sigma(\delta) = \{(\text{cone}_\delta(\tau))^\vee \mid \tau \prec \delta\}$$

is a fan in  $V = N \otimes_{\mathbf{Z}} \mathbf{R}$ , and coincides with the fan consisting of faces of  $\check{\delta}$ . We have

$$X_A(\Sigma(\delta)) = U_{\check{\delta}} = \text{Spec } A[M \cap \delta].$$

Moreover, we have

$$\dim(\text{cone}_\delta(\tau))^\vee = \dim(V) - \dim(\tau)$$

and

$$(-\text{cone}_\delta(\tau)) \cap (\text{cone}_\delta(\tau)) = \tau - \tau.$$

A convex *polytope*  $P$  in  $V = N \otimes_{\mathbf{Z}} \mathbf{R}$  is a subset of  $V$  which can be written as the convex hull of a finite family of element in  $V$ . If this finite family can be chosen to lie in  $N \otimes_{\mathbf{Z}} \mathbf{Q}$ , we say  $P$  is *rational*. If  $P$  is a rational convex polytope in  $V$  such that  $0$  is an interior point of  $P$ , then the set  $\Sigma$  of cones over faces on the boundary of  $P$  is a fan, and the toric scheme  $X_A(\Sigma)$  is proper over  $A$ .

Let  $\Delta$  be a rational convex polytope in the dual space  $V^* = M \otimes_{\mathbf{Z}} \mathbf{R}$ . First consider the case where  $0$  is an interior point of  $\Delta$ . A *face*  $\tau$  of  $\Delta$  is a subset of the form

$$\tau = \{u \in \Delta \mid \langle u, v \rangle = r\},$$

where  $r$  is a real number and  $v \in V$  is a vector such that  $\langle u, v \rangle \geq r$  for all  $u \in \Delta$ . Since  $0$  is an interior point of  $\Delta$ , we have either  $v = 0$  or  $r < 0$ . When  $v = 0$ , the face  $\tau$  is just  $\Delta$ . When  $r < 0$ , we can always choose  $v$  so that  $r = -1$ . So a proper face of  $\Delta$  is of the form

$$\tau = \{u \in \Delta \mid \langle u, v \rangle = -1\},$$

where  $v \in V$  is a vector such that  $\langle u, v \rangle \geq -1$  for all  $u \in \Delta$ .



Define the *polar set*  $\Delta^\circ$  of  $\Delta$  to be

$$\Delta^\circ = \{v \in V \mid \langle u, v \rangle \geq -1 \text{ for all } u \in \Delta\}.$$

It is a rational convex polytope in  $V$  ([F] 1.5, Proposition), and 0 is in its interior. Denote the fan in  $V$  of cones over faces on the boundary of  $\Delta^\circ$  by  $\Sigma(\Delta)$ .

For any face  $\tau$  of  $\Delta$ , let

$$\tau^* = \{v \in \Delta^\circ \mid \langle u, v \rangle = -1 \text{ for all } u \in \tau\}.$$

Then ([F] 1.5, Proposition)  $\tau \mapsto \tau^*$  sets up a one-to-one correspondence between faces of  $\Delta$  and faces of  $\Delta^\circ$ , and

$$\dim(\tau) + \dim(\tau^*) = \dim(V) - 1.$$

The cone  $C(\tau^*)$  in  $V$  over  $\tau^*$  is

$$\begin{aligned} C(\tau^*) &= \{tv \mid t \geq 0, v \in V, \langle u, v \rangle \geq -1 \text{ for all } u \in \Delta, \langle u, v \rangle = -1 \text{ for all } u \in \tau\} \\ &= \{v \mid v \in V, \langle u', v \rangle \geq \langle u, v \rangle \text{ for all } u' \in \Delta, u \in \tau\} \\ &= (\text{cone}_\Delta(\tau))^\vee, \end{aligned}$$

where  $\text{cone}_\Delta(\tau)$  is the cone in  $V^*$  generated by  $u' - u$  ( $u' \in \Delta, u \in \tau$ ). So the fan  $\Sigma(\Delta)$  is the set

$$\Sigma(\Delta) = \{(\text{cone}_\Delta(\tau))^\vee \mid \tau \prec \Delta\}.$$

Note that we have

$$\begin{aligned} \dim((\text{cone}_\Delta(\tau))^\vee) &= \dim(C(\tau^*)) \\ &= \dim(\tau^*) + 1 \\ &= \dim(V) - \dim(\tau). \end{aligned}$$

By [F] 1.2 (10), we have

$$\dim\left((\text{cone}_\Delta(\tau)) \bigcap (-\text{cone}_\Delta(\tau))\right) = \dim(V) - \dim((\text{cone}_\Delta(\tau))^\vee) = \dim(\tau).$$

As

$$\tau - \tau \subset (\text{cone}_\Delta(\tau)) \bigcap (-\text{cone}_\Delta(\tau)),$$

we have

$$(\text{cone}_\Delta(\tau)) \cap (-\text{cone}_\Delta(\tau)) = \text{span}(\tau - \tau).$$

For any  $v \in V$ , the *first meeting locus*  $F_\Delta(v)$  is defined to be the subset of  $\Delta$  where the function  $\langle \cdot, v \rangle|_\Delta$  reaches its minimum. If  $r$  is the minimum of  $\langle \cdot, v \rangle|_\Delta$ , then  $\langle u, v \rangle \geq r$  for all  $u \in \Delta$ , and

$$F_\Delta(v) = \{u \in \Delta | \langle u, v \rangle = r\}.$$

So  $F_\Delta(v)$  is a (nonempty) face of  $\Delta$ . For any subset  $A$  in  $V$ , let

$$F_\Delta(A) = \bigcap_{v \in A} F_\Delta(v).$$

It is also face of  $\Delta$ . (Maybe empty). We have

$$\begin{aligned} C(\tau^*) &= \{v | \langle u', v \rangle \geq \langle u, v \rangle \text{ for all } u' \in \Delta, u \in \tau\} \\ &= \{v | F_\Delta(v) \supset \tau\}. \end{aligned}$$

As  $\tau^*$  is the closure of the complement of its proper faces,  $C(\tau^*)$  is the closure of the set

$$\begin{aligned} &C(\tau^*) - \bigcup_{\tau'^* \prec \tau^*, \tau'^* \neq \tau^*} C(\tau'^*) \\ &= C(\tau^*) - \bigcup_{\tau \prec \tau', \tau \neq \tau'} C(\tau'^*) \\ &= \{v | F_\Delta(v) \supset \tau\} - \bigcup_{\tau \prec \tau', \tau \neq \tau'} \{v | F_\Delta(v) \supset \tau'\} \\ &= \{v | F_\Delta(v) = \tau\}. \end{aligned}$$

Moreover, we have

$$\tau = \bigcap_{v \in C(\tau^*)} F_\Delta(v) = F_\Delta(C(\tau^*)).$$

We summarize these results as follows.

**Proposition 1.2.** Let  $\Delta$  be a rational convex polytope of dimension  $\dim(V)$  in  $V^*$ . For any face  $\tau$  of  $\Delta$ , let  $\text{cone}_\Delta(\tau)$  be the cone generated by  $u' - u$  ( $u' \in \Delta, u \in \tau$ ). Then the family

$$\Sigma(\Delta) = \{(\text{cone}_\Delta(\tau))^\vee | \tau \prec \Delta\}$$

is a fan in  $V = N \otimes_{\mathbf{Z}} \mathbf{R}$ , and  $X_A(\Sigma(\Delta))$  is proper over  $A$ . For any  $v \in V$ , let  $F_\Delta(v)$  be the face of  $\Delta$  where the function  $\langle \cdot, v \rangle|_\Delta$  reaches minimum. Then

$$\begin{aligned} (\text{cone}_\Delta(\tau))^\vee &= \{v | F_\Delta(v) \supset \tau\} \\ &= \overline{\{v | F_\Delta(v) = \tau\}} \end{aligned}$$

The map

$$\tau \mapsto (\text{cone}_\Delta(\tau))^\vee$$

sets up a one-to-one correspondence between faces of  $\Delta$  and cones in the fan  $\Sigma(\Delta)$ , and the inverse map is

$$\sigma \mapsto F_\Delta(\sigma) = \bigcap_{v \in \sigma} F_\Delta(v).$$

Moreover, we have

$$\dim(\text{cone}_\Delta(\tau))^\vee = \dim(V) - \dim(\tau)$$

and

$$(\text{cone}_\Delta(\tau)) \bigcap (-\text{cone}_\Delta(\tau)) = \text{span}(\tau - \tau).$$

Note that Proposition 1.2 holds for the polytope  $\Delta$  if and only if it holds for a translation of  $\Delta$ . Making a translation, we may assume 0 is an interior point of  $\Delta$ . In this case, Proposition 1.2 follows from the previous discussion.

## 2. Extensions of Kummer sheaves to toric schemes

In this section  $k$  is a field. Let  $m$  be a positive integer prime to the characteristic of  $k$  so that  $k$  contains a primitive  $m$ -th root of unity. Let

$$\mu_m(k)^n = \{(\zeta_1, \dots, \zeta_n) \mid \zeta_i \in k, \zeta_i^m = 1\}.$$

The Kummer covering

$$[m] : \mathbf{T}_k^n \rightarrow \mathbf{T}_k^n, x \mapsto x^m$$

on the torus  $\mathbf{T}_k^n$  defines a  $\mu_m(k)^n$ -torsor

$$1 \rightarrow \mu_m(k)^n \rightarrow \mathbf{T}_k^n \xrightarrow{[m]} \mathbf{T}_k^n \rightarrow 1.$$

Let  $\chi : \mu_m(k)^n \rightarrow \overline{\mathbf{Q}}_l^*$  be a character. Pushing-forward the above torsor by  $\chi^{-1}$ , we get a lisse  $\overline{\mathbf{Q}}_l$ -sheaf  $\mathcal{K}_\chi$  on  $\mathbf{T}_k^n$  of rank 1. We call  $\mathcal{K}_\chi$  the *Kummer sheaf* associated to  $\chi$ . The following properties of the Kummer sheaf are standard. We omit their proof.

(a) We have an isomorphism

$$\mathcal{K}_\chi|_{\{1\}} \cong \overline{\mathbf{Q}}_l,$$

where 1 is the identity of the algebraic group  $\mathbf{T}_k^n$ .

(b) Let  $s : \mathbf{T}_k^n \times_k \mathbf{T}_k^n \rightarrow \mathbf{T}_k^n$  be the multiplication on the torus, and let  $\text{pr}_1, \text{pr}_2 : \mathbf{T}_k^n \times_k \mathbf{T}_k^n \rightarrow \mathbf{T}_k^n$  be the two projections. We have an isomorphism

$$s^* \mathcal{K}_\chi \cong \text{pr}_1^* \mathcal{K}_\chi \otimes \text{pr}_2^* \mathcal{K}_\chi$$

which is compatible with the isomorphism

$$(s^* \mathcal{K}_\chi)|_{\{(1,1)\}} \cong (\text{pr}_1^* \mathcal{K}_\chi \otimes \text{pr}_2^* \mathcal{K}_\chi)|_{\{(1,1)\}}$$

defined by (a). Taking the inverse image of the above isomorphism under the morphism

$$\mathbf{T}_k^n \rightarrow \mathbf{T}_k^n \times_k \mathbf{T}_k^n, \quad x \mapsto (x, x^{-1}),$$

we deduce an isomorphism

$$\mathcal{K}_{\chi^{-1}} \cong \mathcal{K}_\chi^\vee.$$

(c) Suppose  $m' | m$  and suppose there exists a character  $\chi' : \mu_{m'}(k)^n \rightarrow \overline{\mathbf{Q}}_l^*$  such that

$$\chi(\zeta) = \chi'(\zeta^{\frac{m}{m'}}).$$

Then we have an isomorphism

$$\mathcal{K}_\chi \cong \mathcal{K}_{\chi'}.$$

(d) Let  $n_i$  ( $i = 1, 2$ ) be positive integers such that  $n = n_1 + n_2$ , let  $\chi_i : \mu_m(k)^{n_i} \rightarrow \overline{\mathbf{Q}}_l^*$  be characters such that

$$\chi(\zeta_1, \dots, \zeta_n) = \chi_1(\zeta_1, \dots, \zeta_{n_1}) \chi_2(\zeta_{n_1+1}, \dots, \zeta_n),$$

and let

$$p_i : \mathbf{T}_k^n = \mathbf{T}_k^{n_1} \times_k \mathbf{T}_k^{n_2} \rightarrow \mathbf{T}_k^{n_i}$$

be the projections. Then we have an isomorphism

$$\mathcal{K}_\chi \cong p_1^* \mathcal{K}_{\chi_1} \otimes p_2^* \mathcal{K}_{\chi_2}.$$

Now let  $N$  be a lattice of rank  $n$ , let  $M = \text{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$  be its dual lattice, let  $V = N \otimes_{\mathbf{Z}} \mathbf{R}$ , and let  $V^* = \text{Hom}_R(V, R) \cong M \otimes_{\mathbf{Z}} \mathbf{R}$ . For any fan  $\Sigma$  in  $V$ , denote the open immersion of the torus  $\mathbf{T}_k^n = \text{Spec } k[M]$  in the toric  $k$ -scheme  $X_k(\Sigma)$  by

$$j : \mathbf{T}_k^n \hookrightarrow X_k(\Sigma).$$

In this section, we study the sheaf  $j_*\mathcal{K}_\chi$  and the perverse sheaf  $j_{!*}(K_\chi[n])$  on  $X_k(\Sigma)$ . (For the definition of  $j_{!*}$ , see [BBD] 1.4.22 and 1.4.24.)

Let  $\delta$  be a rational convex polyhedral cone in  $V^*$  of dimension  $n$ , and let  $\Sigma(\delta)$  be the fan defined in Proposition 1.1. If  $0$  is a face of  $\delta$ , then the map

$$M \cap \delta \rightarrow k, u \mapsto \begin{cases} 1 & \text{if } u = 0, \\ 0 & \text{if } u \neq 0 \end{cases}$$

is a semigroup homomorphism. It induces an epimorphism of  $k$ -algebras  $k[M \cap \delta] \rightarrow k$  and hence a closed immersion

$$x_0 : \text{Spec } k \rightarrow \text{Spec } k[M \cap \delta] = X_k(\Sigma(\delta)).$$

We call  $x_0$  the *distinguished point* in  $X_k(\Sigma(\delta))$ . It is fixed under the torus action on  $X_k(\Sigma(\delta))$ .

**Lemma 2.1.** Let  $\delta$  be a rational convex polyhedral cone in  $V^*$  of dimension  $n$  with  $0$  being a face, let  $x_0 : \text{Spec } k \rightarrow X_k(\Sigma(\delta))$  be the distinguished point in  $X_k(\Sigma(\delta))$ , and let  $j : \mathbf{T}_k^n \rightarrow X_k(\Sigma(\delta))$  be the open immersion of the open dense torus in  $X_k(\Sigma(\delta))$ . If  $\chi : \mu_m(k)^n \rightarrow \overline{\mathbf{Q}}_l^*$  is a nontrivial character, then  $x_0^*(j_{!*}(\mathcal{K}_\chi[n]))$  is acyclic and  $x_0^*(j_*\mathcal{K}_\chi) = 0$ . Moreover, we have  $j_*\overline{\mathbf{Q}}_l = \overline{\mathbf{Q}}_l$ .

**Proof.** Consider the morphism

$$\bar{\epsilon} : \mathbf{T}_k^n \times_k X_k(\Sigma(\delta)) \rightarrow \mathbf{T}_k^n \times_k X_k(\Sigma(\delta)), \bar{\epsilon}(t, x) = (t, tx),$$

where the second component of  $\bar{\epsilon}$  is defined by the action of  $\mathbf{T}_k^n$  on  $X_k(\Sigma(\delta))$ . Note that  $\bar{\epsilon}$  is an isomorphism. Denote the restriction of  $\bar{\epsilon}$  to  $\mathbf{T}_k^n \times_k \mathbf{T}_k^n$  by  $\epsilon$ . Consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{T}_k^n \times_k \mathbf{T}_k^n & \xrightarrow{\epsilon} & \mathbf{T}_k^n \times_k \mathbf{T}_k^n & \xrightarrow{\text{pr}_2} & \mathbf{T}_k^n \\ \downarrow \text{id} \times j & & \downarrow \text{id} \times j & & \downarrow j \\ \mathbf{T}_k^n \times_k X_k(\Sigma(\delta)) & \xrightarrow{\bar{\epsilon}} & \mathbf{T}_k^n \times_k X_k(\Sigma(\delta)) & \xrightarrow{\bar{\text{pr}}_2} & X_k(\Sigma(\delta)), \end{array}$$

where  $\text{pr}_2$  and  $\bar{\text{pr}}_2$  are the projections to the second components. Note that we have

$$\text{pr}_2 \epsilon = s,$$

where  $s$  is the multiplication on the torus. By the smooth base change theorem and the fact that  $\bar{\mathrm{pr}}_2^*[n]$  is an exact functor with respect to the perverse t-structure ([BBD] page 108-109), we have

$$\bar{\mathrm{pr}}_2^*(j_{!*}(\mathcal{K}_\chi[n]))[n] \cong (\mathrm{id} \times j)_{!*}(\mathrm{pr}_2^*\mathcal{K}_\chi[2n]).$$

So we have

$$\begin{aligned} \bar{\epsilon}^* \bar{\mathrm{pr}}_2^*(j_{!*}(\mathcal{K}_\chi[n]))[n] &\cong \bar{\epsilon}^*(\mathrm{id} \times j)_{!*}(\mathrm{pr}_2^*\mathcal{K}_\chi[2n]) \\ &\cong (\mathrm{id} \times j)_{!*}(\epsilon^* \mathrm{pr}_2^*\mathcal{K}_\chi[2n]) \\ &\cong (\mathrm{id} \times j)_{!*}(s^*\mathcal{K}_\chi[2n]) \\ &\cong (\mathrm{id} \times j)_{!*}(\mathrm{pr}_1^*\mathcal{K}_\chi \otimes \mathrm{pr}_2^*\mathcal{K}_\chi[2n]). \end{aligned}$$

Using [BBD] 1.4.24, 4.2.7 and 4.2.8, one can show

$$(\mathrm{id} \times j)_{!*}(\mathrm{pr}_1^*\mathcal{K}_\chi \otimes \mathrm{pr}_2^*\mathcal{K}_\chi[2n]) \cong \mathrm{pr}_1^*(\mathcal{K}_\chi[n]) \otimes \bar{\mathrm{pr}}_2^*(j_{!*}(\mathcal{K}_\chi[n])).$$

So we have

$$\bar{\epsilon}^* \bar{\mathrm{pr}}_2^*(j_{!*}(\mathcal{K}_\chi[n]))[n] \cong \mathrm{pr}_1^*(\mathcal{K}_\chi[n]) \otimes \bar{\mathrm{pr}}_2^*(j_{!*}(\mathcal{K}_\chi[n])).$$

Let  $i$  be the morphism

$$\mathbf{T}_k^n = \mathbf{T}_k^n \times_k \mathrm{Spec} k \xrightarrow{\mathrm{id} \times x_0} \mathbf{T}_k^n \times_k X_k(\Sigma(\delta))$$

and let  $\pi : \mathbf{T}_k^n \rightarrow \mathrm{Spec} k$  be the structure morphism. We have

$$\begin{aligned} i^* \bar{\epsilon}^* \bar{\mathrm{pr}}_2^*(j_{!*}(\mathcal{K}_\chi[n]))[n] &\cong (\bar{\mathrm{pr}}_2 \bar{\epsilon} i)^*(j_{!*}(\mathcal{K}_\chi[n]))[n] \\ &\cong (x_0 \pi)^*(j_{!*}(\mathcal{K}_\chi[n]))[n], \\ i^*(\mathrm{pr}_1^*(\mathcal{K}_\chi[n]) \otimes \bar{\mathrm{pr}}_2^*(j_{!*}(\mathcal{K}_\chi[n]))) &\cong \mathcal{K}_\chi[n] \otimes (x_0 \pi)^*(j_{!*}(\mathcal{K}_\chi[n])). \end{aligned}$$

We thus have an isomorphism

$$(x_0 \pi)^*(j_{!*}(\mathcal{K}_\chi[n])) \cong \mathcal{K}_\chi \otimes (x_0 \pi)^*(j_{!*}(\mathcal{K}_\chi[n])).$$

So for all  $i$ , we have

$$\mathcal{H}^i((x_0 \pi)^*(j_{!*}(\mathcal{K}_\chi[n]))) \cong \mathcal{K}_\chi \otimes \mathcal{H}^i((x_0 \pi)^*(j_{!*}(\mathcal{K}_\chi[n]))).$$

If  $\chi$  is nontrivial, this isomorphism implies that  $\mathcal{H}^i((x_0 \pi)^*(j_{!*}(\mathcal{K}_\chi[n]))) = 0$ . Indeed, we can evaluate  $H^0(\mathbf{T}_k^n, \cdot)$  on both sides of the above isomorphism and use the fact that  $H^0(\mathbf{T}_k^n, \mathcal{K}_\chi) = 0$ .

So we have  $\mathcal{H}^i(x_0^*(j_{!*}(\mathcal{K}_\chi[n]))) = 0$  for all  $i$ , and hence  $x_0^*(j_{!*}(\mathcal{K}_\chi[n]))$  is acyclic.

Similarly, one can show  $x_0^*(j_*\mathcal{K}_\chi) = 0$  if  $\chi$  is nontrivial. To prove  $j_*\overline{\mathbf{Q}}_l = \overline{\mathbf{Q}}_l$ , we need to show for any nonempty connected étale  $X_k(\Sigma(\delta))$ -scheme  $U$ , we have

$$(j_*\overline{\mathbf{Q}}_l)(U) = \overline{\mathbf{Q}}_l,$$

that is,

$$\overline{\mathbf{Q}}_l(U \times_{X_k(\Sigma(\delta))} \mathbf{T}_k^n) = \overline{\mathbf{Q}}_l.$$

It suffices to show  $U \times_{X_k(\Sigma(\delta))} \mathbf{T}_k^n$  is nonempty and connected. Indeed, by [F] 2.1,  $X_k(\Sigma(\delta))$  is normal. So  $U$  is normal. As  $U$  is connected,  $U$  is integral.  $U \times_{X_k(\Sigma(\delta))} \mathbf{T}_k^n$  is an open subscheme of  $U$ . It is nonempty since the image of  $U$  in  $X_k(\Sigma(\delta))$  is nonempty open and hence has nonempty intersection with the open dense torus  $\mathbf{T}_k^n$ . Being a nonempty open subscheme of the integral scheme  $U$ ,  $U \times_{X_k(\Sigma(\delta))} \mathbf{T}_k^n$  is also integral, and in particular connected. This finishes the proof of the lemma.

**Remark 2.2.** Keep the notations in Lemma 2.1. Denote by  $\bar{x}_0$  the geometric point associated to  $x_0$ . By [DL] 6.2.3,  $x_0^*(j_{!*}(\overline{\mathbf{Q}}_l[n]))$  is pure of weight  $n$ , that is, for any  $i$  and any eigenvalue  $\lambda$  of the geometric Frobenius element  $F$  in  $\text{Gal}(\bar{k}/k)$  acting on  $H^i(\bar{x}_0^*(j_{!*}(\overline{\mathbf{Q}}_l[n])))$ ,  $\lambda$  is an algebraic number, and all its galois conjugates have archimedean absolute value  $q^{\frac{i+n}{2}}$ . Moreover, by [DL] 6.2.1,  $\dim(H^i(\bar{x}_0^*(j_{!*}(\overline{\mathbf{Q}}_l[n])))$  coincides with the coefficient of  $T^{i+n}$  of the polynomial  $\alpha(\delta)$  defined in the Introduction.

**Lemma 2.3.** Let  $\Sigma$  be a fan in  $V$ ,  $\sigma \in \Sigma$  and  $\delta$  the image of  $\sigma$  under the canonical homomorphism  $V^* \rightarrow V^*/\sigma^\perp$ . Note that  $\delta$  is a rational convex polyhedral cone in  $V^*/\sigma^\perp$  of dimension  $\dim(V^*/\sigma^\perp)$  and  $0$  is a face of  $\delta$ . Let  $x_0$  be the distinguished point in  $X_k(\Sigma(\delta))$ , and let

$$j : \mathbf{T}_k^n \hookrightarrow X_k(\Sigma), \quad j' : \mathbf{T}_k^{\dim(\sigma)} \hookrightarrow X_k(\Sigma(\delta)) = \text{Spec } k[(M/M \cap \sigma^\perp) \cap \delta]$$

be the immersions of the open dense tori in toric schemes. Denote by

$$p_1 : \mathbf{T}_k^n = \text{Spec } k[M] \rightarrow O_\sigma = \text{Spec } k[M \cap \sigma^\perp]$$

the morphism induced by the canonical homomorphism

$$k[M \cap \sigma^\perp] \hookrightarrow k[M].$$

If  $\mathcal{K}_\chi = p_1^* \mathcal{K}_{\chi_1}$  for some Kummer sheaf  $\mathcal{K}_{\chi_1}$  on  $O_\sigma$ , then

$$\begin{aligned} (j_{!*}(\mathcal{K}_\chi[n]))|_{O_\sigma} &\cong (\mathcal{K}_{\chi_1}[n - \dim(\sigma)] \otimes \pi^* x_0^*(j'_{!*}(\overline{\mathbf{Q}}_l[\dim(\sigma)])), \\ (j_* \mathcal{K}_\chi)|_{O_\sigma} &\cong \mathcal{K}_{\chi_1}, \end{aligned}$$

where  $O_\sigma$  is considered as a subscheme of  $X_k(\Sigma)$  as in §1, and  $\pi : O_\sigma \rightarrow \text{Spec } k$  is the structure morphism. If  $\mathcal{K}_\chi$  is not the inverse image under  $p_1$  of any Kummer sheaf on  $O_\sigma$ , then  $(j_{!*}(\mathcal{K}_\chi[n]))|_{O_\sigma}$  is acyclic and  $(j_* \mathcal{K}_\chi)|_{O_\sigma} = 0$ .

**Proof.** Since  $M/M \cap \sigma^\perp$  is torsion free and hence a free abelian group, the short exact sequence

$$0 \rightarrow M \cap \sigma^\perp \rightarrow M \rightarrow M/M \cap \sigma^\perp \rightarrow 0$$

splits. Fix an isomorphism

$$M \cong M \cap \sigma^\perp \oplus M/M \cap \sigma^\perp.$$

This isomorphism induces identifications

$$\begin{aligned} \check{\sigma} &\cong \sigma^\perp \oplus \delta, \\ M \cap \check{\sigma} &\cong M \cap \sigma^\perp \oplus (M/M \cap \sigma^\perp) \cap \delta, \\ k[M \cap \check{\sigma}] &\cong k[M \cap \sigma^\perp] \otimes_k k[(M/M \cap \sigma^\perp) \cap \delta], \\ U_\sigma = \text{Spec } k[M \cap \check{\sigma}] &\cong \text{Spec } k[M \cap \sigma^\perp] \times_k \text{Spec } k[(M/M \cap \sigma^\perp) \cap \delta] = O_\sigma \times_k X_k(\Sigma(\delta)), \\ \mathbf{T}_k^n = \text{Spec } k[M] &\cong \text{Spec } k[M \cap \sigma^\perp] \times_k \text{Spec } k[M/M \cap \sigma^\perp] = O_\sigma \times_k \mathbf{T}_k^{\dim(\sigma)}. \end{aligned}$$

We can find Kummer sheaves  $\mathcal{K}_{\chi_1}$  on  $O_\sigma$  and  $\mathcal{K}_{\chi_2}$  on  $\mathbf{T}_k^{\dim(\sigma)}$  so that  $\mathcal{K}_\chi$  is identified with

$$p_1^* \mathcal{K}_{\chi_1} \otimes p_2^* \mathcal{K}_{\chi_2}$$

through the isomorphism  $\mathbf{T}_k^n \cong O_\sigma \times_k \mathbf{T}_k^{\dim(\sigma)}$ , where

$$p_1 : O_\sigma \times_k \mathbf{T}_k^{\dim(\sigma)} \rightarrow O_\sigma, \quad p_2 : O_\sigma \times_k \mathbf{T}_k^{\dim(\sigma)} \rightarrow \mathbf{T}_k^{\dim(\sigma)}$$

are the projections. Consider the commutative diagram

$$\begin{array}{ccccccc} O_\sigma & \rightarrow & U_\sigma & \xleftarrow{j} & \mathbf{T}_k^n & & \\ \parallel & & \downarrow \cong & & \downarrow \cong & & \\ O_\sigma & \xrightarrow{i} & O_\sigma \times_k X_k(\Sigma(\delta)) & \xleftarrow{\text{id} \times j'} & O_\sigma \times_k \mathbf{T}_k^{\dim(\sigma)}, & & \end{array}$$



where  $i$  is the morphism

$$O_\sigma = O_\sigma \times_k \text{Spec } k \xrightarrow{\text{id} \times x_0} O_\sigma \times_k X_k(\Sigma(\delta)).$$

Using [BBD] 1.4.24, 4.2.7 and 4.2.8, one can show

$$(\text{id} \times j')_{!*} \left( p_1^* \mathcal{K}_{\chi_1}[n - \dim(\sigma)] \otimes p_2^* \mathcal{K}_{\chi_2}[\dim(\sigma)] \right) \cong \text{pr}_1^* \mathcal{K}_{\chi_1}[n - \dim(\sigma)] \otimes \text{pr}_2^* j'_{!*}(\mathcal{K}_{\chi_2}[\dim(\sigma)]),$$

where

$$\text{pr}_1 : O_\sigma \times_k X_k(\Sigma(\delta)) \rightarrow O_\sigma, \text{pr}_2 : O_\sigma \times_k X_k(\Sigma(\delta)) \rightarrow X_k(\Sigma(\delta))$$

are the projections. So we have

$$\begin{aligned} (j_{!*}(\mathcal{K}_\chi[n]))|_{O_\sigma} &\cong i^*(\text{id} \times j')_{!*} \left( p_1^* \mathcal{K}_{\chi_1}[n - \dim(\sigma)] \otimes p_2^* \mathcal{K}_{\chi_2}[\dim(\sigma)] \right) \\ &\cong i^* \left( \text{pr}_1^* \mathcal{K}_{\chi_1}[n - \dim(\sigma)] \otimes \text{pr}_2^* j'_{!*}(\mathcal{K}_{\chi_2}[\dim(\sigma)]) \right) \\ &\cong \mathcal{K}_{\chi_1}[n - \dim(\sigma)] \otimes \pi^* x_0^*(j'_{!*}(\mathcal{K}_{\chi_2}[\dim(\sigma)])), \end{aligned}$$

that is,

$$(j_{!*}(\mathcal{K}_\chi[n]))|_{O_\sigma} \cong \mathcal{K}_{\chi_1}[n - \dim(\sigma)] \otimes \pi^* x_0^*(j'_{!*}(\mathcal{K}_{\chi_2}[\dim(\sigma)])).$$

Similarly, we have

$$(j_* \mathcal{K}_\chi)|_{O_\sigma} \cong \mathcal{K}_{\chi_1} \otimes \pi^* x_0^*(j'_* \mathcal{K}_{\chi_2}).$$

We then apply Lemma 2.1.

**Proposition 2.4.** Let  $\Sigma$  be a fan in  $V$ ,  $\sigma \in \Sigma$ ,

$$j : \mathbf{T}_k^n \hookrightarrow X_k(\Sigma), j_1 : O_\sigma = \text{Spec } k[M \cap \sigma^\perp] \rightarrow V(\sigma) = X_k(\text{star}(\sigma))$$

the immersions of the open dense tori in toric schemes, and

$$p_1 : \mathbf{T}_k^n = \text{Spec } k[M] \rightarrow O_\sigma = \text{Spec } k[M \cap \sigma^\perp]$$

the morphism induced by the canonical homomorphism

$$k[M \cap \sigma^\perp] \hookrightarrow k[M].$$

If  $\mathcal{K}_\chi = p_1^* \mathcal{K}_{\chi_1}$  for a Kummer sheaf  $\mathcal{K}_{\chi_1}$  on  $O_\sigma$ , then

$$(j_* \mathcal{K}_\chi)|_{V(\sigma)} \cong j_{1*} \mathcal{K}_{\chi_1}.$$

If  $\mathcal{K}_\chi$  is not the inverse image under  $p_1$  of any Kummer sheaf on  $O_\sigma$ , then

$$(j_*\mathcal{K}_\chi)|_{V(\sigma)} = 0.$$

**Proof.** First suppose  $\mathcal{K}_\chi$  is not the inverse image under  $p_1$  of any Kummer sheaf on  $O_\sigma$ . Then for any  $\tau \in \Sigma$  with  $\sigma \prec \tau$ ,  $\mathcal{K}_\chi$  is not the inverse image of any Kummer sheaf on  $O_\tau$ . By Lemma 2.3, we then have  $(j_*\mathcal{K}_\chi)|_{O_\tau} = 0$ . As  $V(\sigma) = \bigcup_{\sigma \prec \tau} O_\tau$ , we have  $(j_*\mathcal{K}_\chi)|_{V(\sigma)} = 0$ .

Now suppose  $\mathcal{K}_\chi = p_1^*\mathcal{K}_{\chi_1}$  for a Kummer sheaf  $\mathcal{K}_1$  on  $O_\sigma$ . As in the proof of Lemma 2.3, we have a commutative diagram

$$\begin{array}{ccc} \mathbf{T}_k^n & \xrightarrow{j_\sigma} & U_\sigma \\ \downarrow \cong & & \downarrow \cong \\ O_\sigma \times_k \mathbf{T}_k^{\dim(\sigma)} & \xrightarrow{\text{id} \times j'} & O_\sigma \times_k X_k(\Sigma(\delta)), \end{array}$$

where  $\delta$  is the image of  $\tilde{\sigma}$  in  $V^*/\sigma^\perp$ , and  $j_\sigma$  and  $j'$  are the open immersions of the tori in  $U_\sigma$  and in  $X_k(\Sigma(\delta))$ , respectively. As  $\mathcal{K}_\chi = p_1^*\mathcal{K}_{\chi_1}$ , the sheaf  $j_{\sigma*}\mathcal{K}_\chi$  can be identified with  $(\text{id} \times j')_*(p^*\mathcal{K}_{\chi_1})$ , where  $p : O_\sigma \times_k \mathbf{T}_k^{\dim(\sigma)} \rightarrow O_\sigma$  is the projection. By [BBD] 4.2.7 (which can be deduced from [SGA 4 $\frac{1}{2}$ ] [Th. finitude] 2.16 and Appendix 2.10), we have

$$(\text{id} \times j')_*(p^*\mathcal{K}_{\chi_1}) \cong \text{pr}_1^*\mathcal{K}_{\chi_1} \otimes \text{pr}_2^*(j'_*\overline{\mathbf{Q}}_l),$$

where

$$\text{pr}_1 : O_\sigma \times_k X_k(\Sigma(\delta)) \rightarrow O_\sigma, \text{pr}_2 : O_\sigma \times_k X_k(\Sigma(\delta)) \rightarrow X_k(\Sigma(\delta))$$

are the projections. As  $j'_*\overline{\mathbf{Q}}_l = \overline{\mathbf{Q}}_l$  by Lemma 2.1, we have

$$(\text{id} \times j')_*(p^*\mathcal{K}_{\chi_1}) \cong \text{pr}_1^*\mathcal{K}_{\chi_1}.$$

It follows that the canonical homomorphism

$$q_\sigma^*\mathcal{K}_{\chi_1} \rightarrow j_{\sigma*}j_\sigma^*q_\sigma^*\mathcal{K}_{\chi_1}$$

is an isomorphism, where  $q_\sigma : U_\sigma \rightarrow O_\sigma$  is the morphism defined by the canonical homomorphism  $k[M \cap \sigma^\perp] \hookrightarrow k[M \cap \tilde{\sigma}]$ .

For each  $\tau \in \Sigma$  with  $\sigma \prec \tau$ , fix notations by the following commutative diagram:

$$\begin{array}{ccccc} & & O_\sigma & \rightarrow & O_\tau \\ & \nearrow p_1 & & & \\ & q_\sigma \uparrow & & & \uparrow q_\tau \\ \mathbf{T}_k^n & \xrightarrow{j_\sigma} & U_\sigma & \xrightarrow{j_{\sigma\tau}} & U_\tau \\ & i_\sigma \uparrow & & & \uparrow i_\tau \\ & O_\sigma & \xrightarrow{l_\tau} & U_\tau(\sigma) = \text{Spec } k[M \cap \sigma^\perp \cap \tilde{\tau}], \end{array}$$

and let  $j_\tau : \mathbf{T}_k^n \hookrightarrow U_\tau$  be the canonical open immersion. If  $\mathcal{K}_\chi$  is the inverse image of a Kummer sheaf  $\mathcal{K}_\tau$  on  $O_\tau$ , then the same argument as above shows that the following canonical homomorphisms are isomorphisms:

$$q_\tau^* \mathcal{K}_{\chi_\tau} \xrightarrow{\cong} j_{\tau*} j_\tau^* q_\tau^* \mathcal{K}_{\chi_\tau} \quad (1)$$

$$(q_\tau i_\tau)^* \mathcal{K}_{\chi_\tau} \xrightarrow{\cong} l_{\tau*} l_\tau^* (q_\tau i_\tau)^* \mathcal{K}_{\chi_\tau} \quad (2)$$

The isomorphism

$$q_\sigma^* \mathcal{K}_{\chi_1} \xrightarrow{\cong} j_{\sigma*} j_\sigma^* q_\sigma^* \mathcal{K}_{\chi_1}$$

induces an isomorphism

$$i_\sigma^* q_\sigma^* \mathcal{K}_{\chi_1} \xrightarrow{\cong} i_\sigma^* j_{\sigma*} j_\sigma^* q_\sigma^* \mathcal{K}_{\chi_1}.$$

So we have an isomorphism

$$\mathcal{K}_{\chi_1} \xrightarrow{\cong} j_{1*} ((j_* \mathcal{K}_\chi)|_{V(\sigma)}) \quad (3)$$

Applying the adjointness of  $(j_1^*, j_{1*})$  to the inverse of this isomorphism, we get a homomorphism

$$(j_* \mathcal{K}_\chi)|_{V(\sigma)} \rightarrow j_{1*} \mathcal{K}_{\chi_1} \quad (4)$$

Let's show this is an isomorphism. This would prove our proposition.

We have  $V(\sigma) = \bigcup_{\sigma \prec \tau} O_\tau$ . It suffices to prove the restriction of the homomorphism (4) to each  $O_\tau$  ( $\sigma \prec \tau$ ) is an isomorphism. If  $\mathcal{K}_\chi$  is not the inverse image of any Kummer sheaf on  $O_\tau$ , the both  $(j_* \mathcal{K}_\chi)|_{O_\tau}$  and  $(j_{1*} \mathcal{K}_{\chi_1})|_{O_\tau}$  vanish by Lemma 2.3, and the restriction to  $O_\tau$  of (4) is trivially an isomorphism. Now suppose  $\mathcal{K}_\chi$  is the inverse image of a Kummer sheaf  $\mathcal{K}_\tau$  on  $O_\tau$ . Let's prove the restriction of (4) to the set  $U_\tau(\sigma)$  (which contains  $O_\tau$ ) is an isomorphism.

Recall that to construct the homomorphism (4), we first construct an isomorphism (3), and then apply the adjointness of  $(j_1^*, j_{1*})$  to the inverse of (3). The isomorphism (3) can also be described as follows: From the isomorphism (1), we get an isomorphism

$$l_\tau^* i_\tau^* q_\tau^* \mathcal{K}_{\chi_\tau} \xrightarrow{\cong} l_\tau^* j_\tau^* j_{\tau*} j_\tau^* q_\tau^* \mathcal{K}_{\chi_\tau} \quad (5)$$

Note that (3) can be identified with (5). This can be seen by applying  $i_\sigma^*$  to the commutative diagram

$$\begin{array}{ccc} j_{\sigma\tau}^* q_\tau^* \mathcal{K}_{\chi_\tau} & \xrightarrow{\cong} & j_{\sigma\tau}^* j_{\tau*} j_\tau^* q_\tau^* \mathcal{K}_{\chi_\tau} \\ \cong \downarrow & & \downarrow \cong \\ q_\sigma^* \mathcal{K}_{\chi_1} & \xrightarrow{\cong} & j_{\sigma*} j_\sigma^* q_\sigma^* \mathcal{K}_{\chi_1}. \end{array}$$

Applying the adjointness of  $(l_\tau^*, l_{\tau*})$  to the inverse of (5), we get

$$i_\tau^* j_{\tau*} j_\tau^* q_\tau^* \mathcal{K}_{\chi_\tau} \rightarrow l_{\tau*} l_\tau^* i_\tau^* q_\tau^* \mathcal{K}_{\chi_\tau} \quad (6)$$

Note that (6) can be identified with the restriction of (4) to  $U_\tau(\sigma)$ . So it suffices to show (6) is an isomorphism. But (6) coincides with the composition

$$i_\tau^* j_{\tau*} j_\tau^* q_\tau^* \mathcal{K}_{\chi_\tau} \rightarrow i_\tau^* q_\tau^* \mathcal{K}_{\chi_\tau} \rightarrow l_{\tau*} l_\tau^* i_\tau^* q_\tau^* \mathcal{K}_{\chi_\tau},$$

where the first arrow is obtained by applying  $i_\tau^*$  to the inverse of (1), and the second arrow can be identified with (2). As (1) and (2) are isomorphisms, (6) is also an isomorphism. This finishes the proof of the proposition.

In the following,  $A$  is a finitely generated  $k$ -algebra. For any fan  $\Sigma$  in  $V$  and any cone  $\sigma$  in  $\Sigma$ , we denote  $\text{Spec } A[M \cap \check{\sigma}]$  (resp.  $\text{Spec } k[M \cap \check{\sigma}]$ ) by  $U_\sigma$  (resp.  $U_{\sigma_k}$ ), and  $\text{Spec } A[M \cap \sigma^\perp]$  (resp.  $\text{Spec } k[M \cap \sigma^\perp]$ ) by  $O_\sigma$  (resp.  $O_{\sigma_k}$ ). They are considered as subschemes of  $X_A(\Sigma)$  (resp.  $X_k(\Sigma)$ .)

**Proposition 2.5.** Let  $A$  be a finitely generated  $k$ -algebra, let  $\Sigma$  be a fan in  $V$  and let  $Y$  be an effective Cartier divisor of  $X_A(\Sigma)$ . Denote by  $j : \mathbf{T}_k^n \rightarrow X_k(\Sigma)$  the immersion of the open dense torus in  $X_k(\Sigma)$ , and denote by  $(j_{!*}(\mathcal{K}_\chi[n]))|_Y$  and  $(j_*\mathcal{K}_\chi)|_Y$  the inverse images under the composition

$$Y \rightarrow X_A(\Sigma) \rightarrow X_k(\Sigma)$$

of  $j_{!*}(\mathcal{K}_\chi[n])$  and  $j_*\mathcal{K}_\chi$ , respectively. Let  $y \in Y$  and let  $\sigma$  be the unique cone in  $\Sigma$  such that  $y \in O_\sigma$ . Suppose the scheme theoretic intersection  $Y \cap O_\sigma$  is smooth over  $A$  at  $y$ , and is not equal to  $O_\sigma$  in any neighborhood of  $y$ . Then  $Y$  is universally locally acyclic over  $A$  at  $y$  relative to  $(j_{!*}(\mathcal{K}_\chi))|_Y$  and relative to  $(j_*\mathcal{K}_\chi)|_Y$ .

**Proof.** As in the proof of Lemma 2.3,  $M \cap \sigma^\perp$  is a direct factor of  $M$ . Choose a sublattice  $M'$  of  $M$  such that

$$M \cong M \cap \sigma^\perp \oplus M'.$$

Let  $\delta$  be the image of  $\check{\sigma}$  in  $M' \otimes_{\mathbf{Z}} \mathbf{R}$ . It is a cone in  $M' \otimes_{\mathbf{Z}} \mathbf{R}$  of dimension  $\dim(\sigma)$  with 0 being a face. We have

$$\check{\sigma} = \sigma^\perp \oplus \delta,$$

$$\begin{aligned}
M \cap \check{\sigma} &= M \cap \sigma^\perp \oplus M' \cap \delta, \\
A[M \cap \check{\sigma}] &\cong A[M \cap \sigma^\perp] \otimes_A A[M' \cap \delta], \\
U_\sigma = \operatorname{Spec} A[M \cap \check{\sigma}] &\cong \operatorname{Spec} A[M \cap \sigma^\perp] \times_A \operatorname{Spec} A[M' \cap \delta] = O_\sigma \times_A \operatorname{Spec} A[M' \cap \delta].
\end{aligned}$$

We have a commutative diagram of Cartesian squares

$$\begin{array}{ccccc}
& & Y \cap U_\sigma & \leftarrow & Y \cap O_\sigma \\
& & \downarrow & & \downarrow \\
\mathcal{O}_\sigma & \leftarrow & U_\sigma & \leftarrow & O_\sigma \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} A & \leftarrow & \operatorname{Spec} A[M' \cap \delta] & \leftarrow & \operatorname{Spec} A,
\end{array}$$

where  $U_\sigma \rightarrow O_\sigma$  is the projection,  $O_\sigma \rightarrow U_\sigma$  is the canonical closed immersion, and  $\operatorname{Spec} A \rightarrow \operatorname{Spec} A[M' \cap \delta]$  is defined by the homomorphism of  $A$ -algebras

$$A[M' \cap \delta] \rightarrow A, \chi^u \mapsto \begin{cases} 1 & \text{if } u = 0, \\ 0 & \text{if } u \neq 0. \end{cases}$$

Choose an open neighborhood  $W$  of  $y$  in  $U_\sigma$  and  $g \in \mathcal{O}_{X_A(\Sigma)}(W)$  so that  $Y \cap W$  is defined by  $g$ . Since  $Y \cap O_\sigma$  is smooth over  $A$  at  $y$  and is not equal to  $O_\sigma$  near  $y$ , by [SGA 1], Exposé II, Théorème 4.10 (iii), the image  $dg$  in  $(\Omega_{O_\sigma/A}^1)_y \otimes_{\mathcal{O}_{O_\sigma, y}} k(y)$  is nonzero. Hence the image of  $dg$  in  $(\Omega_{U_\sigma/A[M' \cap \delta]}^1)_y \otimes_{\mathcal{O}_{U_\sigma, y}} k(y)$  is nonzero. By [SGA 1], Exposé II, Théorème 4.10 (ii), there exists an open neighborhood  $W'$  of  $y$  in  $U_\sigma$  and an étale  $A[M' \cap \delta]$ -morphism

$$W' \rightarrow \operatorname{Spec} A[M' \cap \delta][T_1, \dots, T_r]$$

such that  $Y \cap W'$  is the inverse image of the closed subscheme of  $\operatorname{Spec} A[M' \cap \delta][T_1, \dots, T_r]$  defined by the equation  $T_r = 0$ .

$$\begin{array}{ccc}
Y \cap W' & \rightarrow & \operatorname{Spec} A[M' \cap \delta][T_1, \dots, T_{r-1}] \\
\downarrow & & \downarrow \\
W' & \rightarrow & \operatorname{Spec} A[M' \cap \delta][T_1, \dots, T_r] \\
& \searrow & \downarrow \\
& & \operatorname{Spec} A[M' \cap \delta].
\end{array}$$

We have

$$\mathbf{T}_k^n = \operatorname{Spec} k[M] \cong O_{\sigma_k} \times_k \operatorname{Spec} k[M'].$$

Let

$$p_1 : \mathbf{T}_k^n \rightarrow O_{\sigma_k}, \quad p_2 : \mathbf{T}_k^n \rightarrow \operatorname{Spec} k[M']$$

be the projections. We can find Kummer sheaves  $\mathcal{K}_{\chi_1}$  on  $O_{\sigma_k}$  and  $\mathcal{K}_{\chi_2}$  on  $\operatorname{Spec} k[M']$  so that  $\mathcal{K}_\chi$  is identified with  $p_1^* \mathcal{K}_{\chi_1} \otimes p_2^* \mathcal{K}_{\chi_2}$ . Using [BBD] 1.4.24, 4.2.7 and 4.2.8, one can verify that on  $U_{\sigma_k}$ ,

we have

$$\begin{aligned} j_{!*}(\mathcal{K}_\chi[n]) &\cong \mathrm{pr}_1^*(\mathcal{K}_{\chi_1}[n - \dim(\sigma)]) \otimes \mathrm{pr}_2^*(j'_{!*}(\mathcal{K}_{\chi_2}[\dim(\sigma)])), \\ j_*\mathcal{K}_\chi &\cong \mathrm{pr}_1^*\mathcal{K}_{\chi_1} \otimes \mathrm{pr}_2^*(j'_*\mathcal{K}_{\chi_2}), \end{aligned}$$

where

$$\begin{aligned} \mathrm{pr}_1 : U_{\sigma k} &\cong O_{\sigma k} \times_k \mathrm{Spec} k[M' \cap \delta] \rightarrow O_{\sigma k}, \\ \mathrm{pr}_2 : U_{\sigma k} &\cong O_{\sigma k} \times_k \mathrm{Spec} k[M' \cap \delta] \rightarrow \mathrm{Spec} k[M' \cap \delta] \end{aligned}$$

are the projections, and  $j' : \mathrm{Spec} k[M'] \rightarrow \mathrm{Spec} k[M' \cap \delta]$  is the immersion of the open dense torus in  $\mathrm{Spec} k[M' \cap \delta] = X_k(\Sigma(\delta))$ .

$$\begin{array}{ccc} \mathbf{T}_k^n & \xrightarrow{j} & U_{\sigma k} \\ \downarrow \cong & & \downarrow \cong \\ O_{\sigma k} \times_k \mathrm{Spec} k[M'] & \xrightarrow{\mathrm{id} \times j'} & O_{\sigma k} \times_k \mathrm{Spec} k[M' \cap \delta]. \end{array}$$

By [SGA 4 $\frac{1}{2}$ ] [Th. finitude] Théorème 2.13 (with  $S = \mathrm{Spec} k$ ), the morphism

$$\mathrm{Spec} A[M' \cap \delta][T_1, \dots, T_{r-1}] \rightarrow \mathrm{Spec} A$$

is universally locally acyclic relative to the inverse images of  $j'_{!*}(\mathcal{K}_{\chi_2}[\dim(\sigma)])$  and of  $j'_*\mathcal{K}_{\chi_2}$  under the composition

$$\mathrm{Spec} A[M' \cap \delta][T_1, \dots, T_{r-1}] \rightarrow \mathrm{Spec} k[M' \cap \delta][T_1, \dots, T_{r-1}] \rightarrow \mathrm{Spec} k[M' \cap \delta].$$

As the morphism

$$Y \cap W' \rightarrow \mathrm{Spec} A[M' \cap \delta][T_1, \dots, T_{r-1}]$$

is étale, we see  $Y \cap W'$  is universally locally acyclic over  $A$  relative to the inverse images of  $j'_{!*}(\mathcal{K}_{\chi_2}[\dim(\sigma)])$  and of  $j'_*\mathcal{K}_{\chi_2}$  under the composition

$$Y \cap W' \rightarrow \mathrm{Spec} A[M' \cap \delta] \rightarrow \mathrm{Spec} k[M' \cap \delta].$$

Note that these inverse images coincides with the inverse images of  $\mathrm{pr}_2^*(j'_{!*}(\mathcal{K}_{\chi_2}[\dim(\sigma)]))$  and of  $\mathrm{pr}_2^*(j'_*\mathcal{K}_{\chi_2})$  under the composition

$$Y \cap W' \rightarrow W' \rightarrow U_\sigma \rightarrow U_{\sigma k}.$$

By Lemma 2.6 below,  $Y \cap W'$  is universally locally acyclic over  $A$  relative to the inverse images of  $\mathrm{pr}_1^*(\mathcal{K}_{\chi_1}[n - \dim(\sigma)]) \otimes \mathrm{pr}_2^*(j'_! (\mathcal{K}_{\chi_2}[\dim(\sigma)]))$  and of  $\mathrm{pr}_1^* \mathcal{K}_{\chi_1} \otimes \mathrm{pr}_2^*(j'_* \mathcal{K}_{\chi_2})$ . Therefore  $Y \cap W'$  is universally locally acyclic over  $A$  relative to the inverse images of  $j_{!*}(\mathcal{K}_{\chi}[n])$  and of  $j_* \mathcal{K}_{\chi}$ . This proves our assertion.

**Lemma 2.6.** Let  $f : X \rightarrow S$  be a morphism,  $\mathcal{K}$  an object in the derived category  $D^+(X, \mathbf{Z}/l^\alpha)$  of étale sheaves of  $(\mathbf{Z}/l^\alpha)$ -modules, and  $\mathcal{F}$  a flat locally constant sheaf of  $(\mathbf{Z}/l^\alpha)$ -modules. Suppose  $f$  is locally acyclic relative to  $\mathcal{K}$ . Then  $f$  is locally acyclic relative to  $\mathcal{F} \otimes \mathcal{K}$ .

**Proof.** Let  $\bar{s}$  be an arbitrary geometric point in  $S$ , let  $\tilde{S}_{\bar{s}}$  be the strict henselization of  $S$  at  $\bar{s}$ . and let  $\bar{t}$  be an arbitrary geometric point in  $\tilde{S}_{\bar{s}}$ . Fix notations by the following commutative diagram of Cartesian squares

$$\begin{array}{ccccc} X_{\bar{t}} & \xrightarrow{j} & X \times_S \tilde{S}_{\bar{s}} & \xleftarrow{i} & X_{\bar{s}} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{t} & \rightarrow & \tilde{S}_{\bar{s}} & \leftarrow & \bar{s}. \end{array}$$

Since  $f$  is locally acyclic relative to  $\mathcal{K}$ , the canonical morphism

$$i^*(\mathcal{K}|_{X \times_S \tilde{S}_{\bar{s}}}) \rightarrow i^* Rj_* j^*(\mathcal{K}|_{X \times_S \tilde{S}_{\bar{s}}})$$

is an isomorphism. Since  $\mathcal{F}$  is flat, we have an isomorphism

$$i^*(\mathcal{F}|_{X \times_S \tilde{S}_{\bar{s}}}) \otimes i^*(\mathcal{K}|_{X \times_S \tilde{S}_{\bar{s}}}) \xrightarrow{\cong} i^*(\mathcal{F}|_{X \times_S \tilde{S}_{\bar{s}}}) \otimes i^* Rj_* j^*(\mathcal{K}|_{X \times_S \tilde{S}_{\bar{s}}}),$$

that is

$$i^*((\mathcal{F} \otimes \mathcal{K})|_{X \times_S \tilde{S}_{\bar{s}}}) \cong i^*(\mathcal{F}|_{X \times_S \tilde{S}_{\bar{s}}} \otimes Rj_* j^*(\mathcal{K}|_{X \times_S \tilde{S}_{\bar{s}}}).$$

Since  $\mathcal{F}$  is flat and locally constant, by [SGA 4] Exposé XVII 5.2.11.1, we have

$$\begin{aligned} \mathcal{F}|_{X \times_S \tilde{S}_{\bar{s}}} \otimes Rj_* j^*(\mathcal{K}|_{X \times_S \tilde{S}_{\bar{s}}}) &\cong Rj_*(j^*(\mathcal{F}|_{X \times_S \tilde{S}_{\bar{s}}}) \otimes j^*(\mathcal{K}|_{X \times_S \tilde{S}_{\bar{s}}})) \\ &\cong Rj_* j^*((\mathcal{F} \otimes \mathcal{K})|_{X \times_S \tilde{S}_{\bar{s}}}). \end{aligned}$$

So the canonical morphism

$$i^*((\mathcal{F} \otimes \mathcal{K})|_{X \times_S \tilde{S}_{\bar{s}}}) \rightarrow i^* Rj_* j^*((\mathcal{F} \otimes \mathcal{K})|_{X \times_S \tilde{S}_{\bar{s}}})$$

is an isomorphism. Therefore  $\mathcal{F} \otimes \mathcal{K}$  is locally acyclic.

**Proposition 2.7.** Let  $A$  be a finitely generated  $k$ -algebra, let  $\Sigma$  be a fan in  $V$ , let  $Y$  be an effective Cartier divisor of  $X_A(\Sigma)$ , and let  $j_Y : Y \cap \mathbf{T}_A^n \rightarrow Y$  be the open immersion induced by the open immersion of the open dense torus in  $X_A(\Sigma)$ . Suppose  $A$  is an integral domain of dimension  $d$  and is smooth over  $k$ , and suppose for any  $\sigma \in \Sigma$ , the scheme theoretic intersection  $Y \cap O_\sigma$  is smooth over  $k$  and is not equal to  $O_\sigma$  near any point. (In particular, taking  $\sigma = 0$ , we see  $Y \cap \mathbf{T}_A^n$  is smooth over  $k$  and is not equal to  $\mathbf{T}_A^n$  anywhere). Then we have

$$\begin{aligned} j_{Y!}(\mathcal{K}_\chi|_{Y \cap \mathbf{T}_A^n}[n+d-1]) &\cong (j_{!}(\mathcal{K}_\chi[n]))|_Y[d-1], \\ j_{Y*}(\mathcal{K}_\chi|_{Y \cap \mathbf{T}_A^n}) &\cong (j_*\mathcal{K}_\chi)|_Y, \end{aligned}$$

where  $(j_{!}(\mathcal{K}_\chi[n]))|_Y$  and  $(j_*\mathcal{K}_\chi)|_Y$  are the same as in Proposition 2.5, and  $\mathcal{K}_\chi|_{Y \cap \mathbf{T}_A^n}$  is the inverse image of  $\mathcal{K}_\chi$  under the composition

$$Y \cap \mathbf{T}_A^n \rightarrow \mathbf{T}_A^n \rightarrow \mathbf{T}_k^n.$$

**Proof.** Let  $y$  be an arbitrary point in  $Y$ . Choose  $\sigma \in \Sigma$  such that  $y \in O_\sigma$ . Choose a sublattice  $M'$  of  $M$  such that

$$M \cong M \cap \sigma^\perp \oplus M',$$

and let  $\delta$  be the image of  $\tilde{\sigma}$  in  $M' \otimes_{\mathbf{Z}} \mathbf{R}$ . Again we have

$$U_\sigma = \text{Spec } A[M \cap \tilde{\sigma}] \cong \text{Spec } A[M \cap \sigma^\perp] \times_A \text{Spec } A[M' \cap \delta] = O_\sigma \times_A \text{Spec } A[M' \cap \delta],$$

and we have a commutative diagram of Cartesian squares

$$\begin{array}{ccccc} & & Y \cap U_\sigma & \leftarrow & Y \cap O_\sigma \\ & & \downarrow & & \downarrow \\ \mathcal{O}_\sigma & \leftarrow & U_\sigma & \leftarrow & O_\sigma \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } A & \leftarrow & \text{Spec } A[M' \cap \delta] & \leftarrow & \text{Spec } A \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } k & \leftarrow & \text{Spec } k[M' \cap \delta] & \leftarrow & \text{Spec } k. \end{array}$$

Choose an open neighborhood  $W$  of  $y$  in  $U_\sigma$  and  $g \in \mathcal{O}_{X_A(\Sigma)}(W)$  so that  $Y \cap W$  is defined by  $g$ . Since  $A$  is a smooth  $k$ -algebra,  $O_\sigma = \text{Spec } A[M \cap \sigma^\perp]$  is smooth over  $k$ . Since  $Y \cap O_\sigma$  is smooth over  $k$  at  $y$  and is not equal to  $O_\sigma$  near  $y$ , by [SGA 1] Exposé II, Théorème 4.10 (iii), the image of  $dg$  in  $(\Omega_{O_\sigma/k}^1)_y \otimes_{\mathcal{O}_{O_\sigma,y}} k(y)$  is nonzero. Hence the image of  $dg$  in  $(\Omega_{U_\sigma/k[M' \cap \delta]}^1)_y \otimes_{\mathcal{O}_{U_\sigma,y}} k(y)$  is nonzero. By [SGA 1] Exposé II, Théorème 4.10 (i), there exists an open neighborhood  $W'$  of  $y$  in



$U_\sigma$  so that  $Y \cap W'$  is smooth over  $k[M' \cap \delta]$ . Choose Kummer sheaves  $\mathcal{K}_{\chi_1}$  on  $O_{\sigma k}$  and  $\mathcal{K}_{\chi_2}$  on  $\text{Spec } k[M']$  so that  $\mathcal{K}_\chi$  is identified with  $p_1^* \mathcal{K}_{\chi_1} \otimes p_2^* \mathcal{K}_{\chi_2}$ , where

$$p_1 : \mathbf{T}_k^n \rightarrow O_{\sigma k}, \quad p_2 : \mathbf{T}_k^n \rightarrow \text{Spec } k[M']$$

are the projections. As in the proof of Proposition 2.5, on  $U_{\sigma k}$ , we have

$$\begin{aligned} j_{!*}(\mathcal{K}_\chi[n]) &\cong \text{pr}_1^*(\mathcal{K}_{\chi_1}[n - \dim(\sigma)]) \otimes \text{pr}_2^*(j'_{!*}(\mathcal{K}_{\chi_2}[\dim(\sigma)])), \\ j_* \mathcal{K}_\chi &= \text{pr}_1^* \mathcal{K}_{\chi_1} \otimes \text{pr}_2^*(j'_* \mathcal{K}_{\chi_2}), \end{aligned}$$

where

$$\begin{aligned} \text{pr}_1 : U_{\sigma k} &\cong O_{\sigma k} \times_k \text{Spec } k[M' \cap \delta] \rightarrow O_{\sigma k}, \\ \text{pr}_2 : U_{\sigma k} &\cong O_{\sigma k} \times_k \text{Spec } k[M' \cap \delta] \rightarrow \text{Spec } k[M' \cap \delta] \end{aligned}$$

are the projections, and  $j' : \text{Spec } k[M'] \rightarrow \text{Spec } k[M' \cap \delta]$  is the immersion of the open dense torus in  $\text{Spec } k[M' \cap \delta] = X_k(\Sigma(\delta))$ . We have the following commutative diagram of Cartesian squares

$$\begin{array}{ccccc} Y \cap W' \cap \mathbf{T}_A^n & \xrightarrow{j_Y} & Y \cap W' & & \\ \downarrow & & \downarrow & & \\ \mathbf{T}_A^n = \text{Spec } A[M] & \hookrightarrow & U_\sigma = \text{Spec } A[M \cap \sigma] & & \\ \downarrow & & \downarrow & & \\ \mathbf{T}_k^n = \text{Spec } k[M] & \hookrightarrow & U_{\sigma k} = \text{Spec } k[M \cap \sigma] & \xrightarrow{\text{pr}_1} & O_{\sigma k} \\ p_2 \downarrow & & \downarrow \text{pr}_2 & & \downarrow \\ \text{Spec } k[M'] & \xrightarrow{j'} & \text{Spec } k[M' \cap \delta] & \rightarrow & \text{Spec } k. \end{array}$$

Since  $Y \cap W'$  is smooth over  $\text{Spec } k[M' \cap \delta]$ , by the smooth base change theorem, on  $Y \cap W'$ , we have

$$(\text{pr}_2^*(j'_* \mathcal{K}_{\chi_2}))|_{Y \cap W'} \cong j_{Y*}((p_2^* \mathcal{K}_{\chi_2})|_{Y \cap \mathbf{T}_A^n})$$

So we have

$$\begin{aligned} (j_* \mathcal{K}_\chi)|_{Y \cap W'} &\cong (\text{pr}_1^* \mathcal{K}_{\chi_1} \otimes \text{pr}_2^*(j'_* \mathcal{K}_{\chi_2}))|_{Y \cap W'} \\ &\cong (\text{pr}_1^* \mathcal{K}_{\chi_1})|_{Y \cap W'} \otimes j_{Y*}((p_2^* \mathcal{K}_{\chi_2})|_{Y \cap \mathbf{T}_A^n}) \\ &\cong j_{Y*}(j_Y^*((\text{pr}_1^* \mathcal{K}_{\chi_1})|_{Y \cap W'}) \otimes (p_2^* \mathcal{K}_{\chi_2})|_{Y \cap \mathbf{T}_A^n}) \\ &\cong j_{Y*}((p_1^* \mathcal{K}_{\chi_1} \otimes p_2^* \mathcal{K}_{\chi_2})|_{Y \cap \mathbf{T}_A^n}) \\ &\cong j_{Y*}(\mathcal{K}_\chi|_{Y \cap \mathbf{T}_A^n}), \end{aligned}$$

where for the third isomorphism, we use [SGA 4] Exposé XVII, 5.2.11.1. This proves that

$$j_{Y*}(\mathcal{K}_\chi|_{Y \cap \mathbf{T}_A^n}) \cong (j_*\mathcal{K}_\chi)|_Y$$

holds when restricted to  $Y \cap W'$ . As  $W'$  is a neighborhood of  $y$ , and  $y$  can be taken to be any point in  $Y$ , we have the above isomorphism everywhere on  $Y$ . Denote the smooth morphism  $Y \cap W' \rightarrow \operatorname{Spec} k[M' \cap \delta]$  by  $f$ . Its relative dimension is  $n - \dim(\sigma) + d - 1$ . Using the smooth base change theorem and the fact that  $f^*[n - \dim(\sigma) + d - 1]$  is exact with respect to the perverse t-structure ([BBD] page 108-109), one can prove the assertion about  $j_{!*}$ .

### 3. Compactification of $f : \mathbf{T}_k^n \rightarrow \mathbf{A}_k^1$

In this section, we take  $N$  to be the lattice  $\mathbf{Z}^n$  and  $V = \mathbf{R}^n$ . Let

$$G = \sum_{i \in \mathbf{Z}^n} a_i X^i \in A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$$

be a Laurent polynomial with coefficients in a commutative ring  $A$ . The *Newton polyhedron*  $\Delta(G)$  of  $G$  is the convex hull in  $\mathbf{R}^n$  of the set  $\{i \in \mathbf{Z}^n | a_i \neq 0\}$ . For any face  $\tau$  of  $\Delta(G)$ , set

$$G_\tau = \sum_{i \in \tau} a_i X^i.$$

Assume  $\dim(\Delta(G)) = n$ , and let  $\Sigma$  be a fan which is a subdivision of the fan  $\Sigma(\Delta(G))$  in Proposition 1.2. Let  $Y$  be the scheme theoretic closure in the toric scheme  $X_A(\Sigma)$  of the locus  $G = 0$  in the torus  $\mathbf{T}_A^n = \operatorname{Spec} A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ . Here we regard the torus  $\mathbf{T}_A^n$  as an open subscheme of  $X_A(\Sigma)$ . Note that  $X_A(\Sigma)$  and  $Y$  are proper over  $A$ .

**Proposition 3.1.** Notation as above. Suppose  $A$  is an integral domain. Let  $\sigma \in \Sigma$  and let  $\tau = F_{\Delta(G)}(\sigma)$ . Then

$$(\tau \cap \{i \in \mathbf{Z}^n | a_i \neq 0\}) \neq \emptyset.$$

Let  $P \in \tau \cap \{i \in \mathbf{Z}^n | a_i \neq 0\}$ . Then we have  $X^{-P}G \in A[\mathbf{Z}^n \cap \check{\sigma}]$  and  $X^{-P}G_\tau \in A[\mathbf{Z}^n \cap \sigma^\perp]$ . Moreover, on the open subscheme  $U_\sigma = \operatorname{Spec} A[\mathbf{Z}^n \cap \check{\sigma}]$  of  $X_A(\Sigma)$ ,  $Y \cap U_\sigma$  is the locus of  $X^{-P}G = 0$ , and on the torus  $O_\sigma = \operatorname{Spec} A[\mathbf{Z}^n \cap \sigma^\perp]$  (regarded as a subscheme of  $X_A(\Sigma)$ ),  $Y \cap O_\sigma$  is the locus of  $X^{-P}G_\tau = 0$ .

**Proof.** Since  $\Sigma$  is a subdivision of  $\Sigma(\Delta(G))$ , we can choose  $\sigma' \in \Sigma(\Delta(G))$  such that  $\sigma \subset \sigma'$ . By the construction in Proposition 1.2,  $F_{\Delta(G)}(\sigma')$  is not empty. As  $\tau = F_{\Delta(G)}(\sigma) \supset F_{\Delta(G)}(\sigma')$ ,  $\tau$  is nonempty. Since  $\tau$  is a face of  $\Delta(G)$ , we must have  $(\tau \cap \{i \in \mathbf{Z}^n | a_i \neq 0\}) \neq \emptyset$  by the definition of  $\Delta(G)$ .

For any  $v \in \sigma$ , we have

$$F_{\Delta(G)}(v) \supset F_{\Delta(G)}(\sigma) \supset \tau.$$

So  $\langle u', v \rangle \geq \langle u, v \rangle$  for any  $u' \in \Delta(G)$  and  $u \in \tau$ . Hence  $v \in (\text{cone}_{\Delta(G)}(\tau))^\vee$ . So  $\sigma \subset (\text{cone}_{\Delta(G)}(\tau))^\vee$  and hence  $\text{cone}_{\Delta(G)}(\tau) \subset \check{\sigma}$ . In particular, we have  $X^{-P}G \in A[\mathbf{Z}^n \cap \check{\sigma}]$ . Since

$$\tau - \tau \subset (\text{cone}_{\Delta(G)}(\tau)) \cap (-\text{cone}_{\Delta(G)}(\tau)) \subset \check{\sigma} \cap (-\check{\sigma}) = \sigma^\perp,$$

we have  $X^{-P}G_\tau \in A[\mathbf{Z}^n \cap \sigma^\perp]$ . For any  $Q \in \Delta(G)$  with  $Q \not\subset \tau = F_{\Delta(G)}(\sigma)$ , we have  $Q \not\subset F_{\Delta(G)}(v)$  for some  $v \in \sigma$ . By the definition of  $F_{\Delta(G)}(v)$ , we must have  $\langle Q, v \rangle > \langle P, v \rangle$ . So  $Q - P \notin \sigma^\perp$ . Therefore, under the epimorphism

$$A[\mathbf{Z}^n \cup \check{\sigma}] \rightarrow A[\mathbf{Z}^n \cup \sigma^\perp], \chi^u \mapsto \begin{cases} \chi^u & \text{if } u \in \sigma^\perp, \\ 0 & \text{if } u \notin \sigma^\perp, \end{cases}$$

$X^{-P}G$  is mapped to  $X^{-P}G_\tau$ . As this epimorphism defines the immersion  $O_\sigma \rightarrow U_\sigma$ , if we can prove  $Y \cap U_\sigma$  is the locus of  $X^{-P}G = 0$  in  $U_\sigma$ , then  $Y \cap O_\sigma$  is the locus of  $X^{-P}G_\tau = 0$  in  $O_\sigma$ .

Since  $Y$  is the scheme theoretic closure in  $X_A(\Sigma)$  of the locus  $G = 0$  in  $\mathbf{T}_A^n = \text{Spec } A[\mathbf{Z}^n]$ , to prove  $Y \cap U_\sigma$  is the locus of  $X^{-P}G = 0$  in  $U_\sigma$ , we need to show the kernel of the composition

$$A[\mathbf{Z}^n \cup \check{\sigma}] \rightarrow A[\mathbf{Z}^n] \rightarrow A[\mathbf{Z}^n]/(G) \tag{1}$$

is the ideal generated by  $X^{-P}G$ . Let  $\{v_1, \dots, v_k\}$  be a minimal family of generators of  $\sigma$ , and let  $\sigma_i$  ( $i = 1, \dots, k$ ) be the rays generated by  $v_i$ . Note that  $\sigma_i$  are one dimensional faces of  $\sigma$ . We have  $\check{\sigma} = \bigcap_{i=1}^k \check{\sigma}_i$  and hence

$$A[\mathbf{Z}^n \cap \check{\sigma}] = \bigcap_{i=1}^k A[\mathbf{Z}^n \cap \check{\sigma}_i].$$

It suffices to show the kernel of the composition

$$A[\mathbf{Z}^n \cap \check{\sigma}_i] \rightarrow A[\mathbf{Z}^n] \rightarrow A[\mathbf{Z}^n]/(G) \tag{2}$$

is the ideal generated by  $X^{-P}G$  for each  $i$ . Indeed, if this is true and if  $f$  lies in the kernel of the composition (1), then  $f$  lies in the kernel of the composition (2) for each  $i$ . Hence  $f$  lies in the

ideal of  $A[\mathbf{Z}^n \cap \check{\sigma}_i]$  generated by  $X^{-P}G$  for each  $i$ . So

$$\frac{f}{X^{-P}G} \in \bigcap_{i=1}^k A[\mathbf{Z}^n \cap \check{\sigma}_i] = A[\mathbf{Z}^n \cap \check{\sigma}],$$

that is,  $f$  lies in the ideal of  $A[\mathbf{Z}^n \cap \check{\sigma}]$  generated by  $X^{-P}G$ .

For each  $i$ ,  $\mathbf{Z}^n / (\mathbf{Z}^n \cap \text{span}(\sigma_i))$  has no torsion. So  $\mathbf{Z}^n \cap \text{span}(\sigma_i)$  is a direct factor of  $\mathbf{Z}^n$ . Since  $\sigma_i$  is a one-dimensional rational cone, we can choose a basis  $e_1, \dots, e_n$  of  $\mathbf{Z}^n$  so that  $\sigma_i$  is generated by  $e_1$ . Denote by  $Y_1, \dots, Y_n$  the coordinates with respect to this basis. We then have isomorphisms

$$\begin{aligned} A[\mathbf{Z}^n \cap \check{\sigma}_i] &\cong A[Y_1, Y_2, Y_2^{-1}, \dots, Y_n, Y_n^{-1}], \\ A[\mathbf{Z}^n] &\cong A[Y_1, Y_1^{-1}, Y_2, Y_2^{-1}, \dots, Y_n, Y_n^{-1}]. \end{aligned}$$

Through these isomorphisms, the open immersion  $\mathbf{T}_A^n \hookrightarrow U_{\sigma_i}$  corresponds to the canonical homomorphism  $A[Y_1, Y_2, Y_2^{-1}, \dots, Y_n, Y_n^{-1}] \hookrightarrow A[Y_1, Y_1^{-1}, Y_2, Y_2^{-1}, \dots, Y_n, Y_n^{-1}]$ . We need to show the kernel of the composition

$$\begin{aligned} A[Y_1, Y_2, Y_2^{-1}, \dots, Y_n, Y_n^{-1}] &\hookrightarrow A[Y_1, Y_1^{-1}, Y_2, Y_2^{-1}, \dots, Y_n, Y_n^{-1}] \\ &\rightarrow A[Y_1, Y_1^{-1}, Y_2, Y_2^{-1}, \dots, Y_n, Y_n^{-1}] / (G) \\ &= A[Y_1, Y_1^{-1}, Y_2, Y_2^{-1}, \dots, Y_n, Y_n^{-1}] / (X^{-P}G) \end{aligned}$$

is the ideal of  $A[Y_1, Y_2, Y_2^{-1}, \dots, Y_n, Y_n^{-1}]$  generated by  $X^{-P}G$ . Let  $B = A[Y_2, Y_2^{-1}, \dots, Y_n, Y_n^{-1}]$ . If  $f$  lies in the kernel of the above composition, then  $f = X^{-P}Gg$  for some  $g \in B[Y_1, Y_1^{-1}]$ . Note that both  $f$  and  $X^{-P}G$  are in  $B[Y_1]$ . By the choice of  $P$ ,  $X^{-P}G$  has a nonzero constant term. This implies that  $g \in B[Y_1]$ . So  $f$  lies in the ideal of  $B[Y_1]$  generated by  $X^{-P}G$ . This finishes the proof of the proposition.

Let

$$f = \sum_{i \in \mathbf{Z}^n} a_i X^i \in A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$$

be a Laurent polynomial. Recall that the Newton polyhedron  $\Delta_\infty(f)$  of  $f$  at  $\infty$  is the convex hull in  $\mathbf{R}^n$  of the set  $\{i \in \mathbf{Z}^n | a_i \neq 0\} \cup \{0\}$ . We say  $f$  is *non-degenerate* with respect to  $\Delta_\infty(f)$  if for any face  $\tau$  of  $\Delta_\infty(f)$  not containing 0, the locus of

$$\frac{\partial f_\tau}{\partial X_1} = \dots = \frac{\partial f_\tau}{\partial X_n} = 0$$

in  $\mathbf{T}_A^n$  is empty. By [SGA 1] Exposé II, Corollaire 4.5, this is equivalent to saying that the morphism

$$f_\tau : \mathbf{T}_A^n = \operatorname{Spec} A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] \rightarrow \mathbf{A}_A^1 = \operatorname{Spec} A[T]$$

defined by the  $A$ -algebra homomorphism

$$A[T] \rightarrow A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}], \quad T \mapsto f_\tau$$

is smooth.

Let  $k$  be a field, and let  $f = \sum_{i \in \mathbf{Z}^n} a_i X^i \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  be a Laurent polynomial with coefficients in  $k$ . It defines a  $k$ -morphism

$$f : \mathbf{T}_k^n \rightarrow \mathbf{A}_k^1.$$

Let

$$A = k[T]$$

and let

$$G = f - T.$$

Regard  $G$  as a Laurent polynomial over  $A$ . We have

$$\Delta(G) = \Delta_\infty(f).$$

Suppose  $\dim(\Delta_\infty(f)) = n$ . Let  $\Sigma$  be a fan that is a subdivision of  $\Sigma(\Delta_\infty(f))$ , let  $Y$  be the scheme theoretic closure in  $X_A(\Sigma)$  of the locus  $G = 0$  in  $\mathbf{T}_A^n$ , and let  $g : Y \rightarrow \mathbf{A}_k^1$  be the composition  $Y \rightarrow X_A(\Sigma) \rightarrow \operatorname{Spec} A$ . Note that  $g$  is proper. The locus of  $G = 0$  in  $\mathbf{T}_A^n$  is the closed subscheme  $\operatorname{Spec} k[T, X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]/(f - T)$  of  $\mathbf{T}_A^n = \operatorname{Spec} k[T, X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ . Since we have an isomorphism

$$k[T, X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]/(f - T) \xrightarrow{\cong} k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}], \quad T \mapsto f,$$

the locus of  $G = 0$  in  $\mathbf{T}_A^n$  can be identified with  $\mathbf{T}_k^n = \operatorname{Spec} k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  and the restriction of  $g$  to this locus can be identified with  $f : \mathbf{T}_k^n \rightarrow \mathbf{A}_k^1$ . So  $g$  is a compactification of  $f$ .

Let

$$j_Y : \mathbf{T}_k^n \cong Y \cap \mathbf{T}_A^n \rightarrow Y$$

be the open immersion induced by the immersion of the open dense torus  $\mathbf{T}_A^n$  in  $X_A(\Sigma)$ , and let  $\mathcal{K}_\chi$  be a Kummer sheaf on  $\mathbf{T}_k^n$ . In the following, we study the properties of  $j_{Y*}\mathcal{K}_\chi$  and  $j_{Y!}(\mathcal{K}_\chi[n])$ . We start with a lemma.

**Lemma 3.2.** Let

$$f = \sum_{i \in \mathbf{Z}^n} a_i X^i \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$$

be a Laurent polynomial over a field  $k$ . Suppose  $\dim(\Delta_\infty(f)) = n$ . Let  $\Sigma$  be a subdivision of  $\Sigma(\Delta_\infty(f))$ ,  $A = k[T]$ ,  $G = f - T$ ,  $Y$  the scheme theoretic closure in  $X_A(\Sigma)$  of the locus  $G = 0$  in  $\mathbf{T}_A^n$ , and  $g : Y \rightarrow \mathbf{A}_k^1$  the composition  $Y \rightarrow X_A(\Sigma) \rightarrow \text{Spec } A$ . Let  $\sigma \in \Sigma$  and let  $\tau = F_{\Delta_\infty(f)}(\sigma)$ .

(i)  $Y \cap O_\sigma$  is not equal to  $O_\sigma$  near any point.

(ii) Suppose  $0 \notin \tau$ . Choose  $P \in \tau \cap \{i \in \mathbf{Z}^n | a_i \neq 0\}$  and let  $Z$  be the locus of  $X^{-P}f_\tau = 0$  in  $O_{\sigma k} = \text{Spec } k[\mathbf{Z}^n \cap \sigma^\perp]$ . Then we have a Cartesian diagram

$$\begin{array}{ccc} Y \cap O_\sigma & \rightarrow & Z \\ g \downarrow & & \downarrow \\ \mathbf{A}_k^1 & \rightarrow & \text{Spec } k. \end{array}$$

If  $f$  is non-degenerate, then  $Z$  is smooth over  $k$ , and hence  $Y \cap O_\sigma$  is smooth over  $\mathbf{A}_k^1$ , and over  $k$ .

(iii) Suppose  $0 \in \tau$ . Then we have an isomorphism

$$Y \cap O_\sigma \cong O_{\sigma k}$$

and  $g|_{Y \cap O_\sigma}$  is identified with the morphism  $f_\tau : O_{\sigma k} \rightarrow \mathbf{A}_k^1$  defined by the  $k$ -algebra homomorphism

$$k[T] \rightarrow k[\mathbf{Z}^n \cap \sigma^\perp], \quad T \mapsto f_\tau.$$

In particular,  $Y \cap O_\sigma$  is smooth over  $k$ . If  $f$  is non-degenerate and  $\sigma \in \Sigma(\Delta_\infty(f))$ , then  $Y \cap O_\sigma$  is smooth over  $\mathbf{A}_k^1$  outside finitely many closed points.

**Proof.**

(i) Let  $P \in \tau \cap (\{i \in \mathbf{Z}^n | a_i \neq 0\} \cup \{0\})$ . By Proposition 3.1,  $Y \cap O_\sigma$  is the locus of  $X^{-P}G_\tau = 0$  in  $O_\sigma = \text{Spec } A[\mathbf{Z}^n \cap \sigma^\perp]$ . Since  $X^{-P}G_\tau$  is nonzero and  $O_\sigma$  is integral,  $Y \cap O_\sigma$  is not equal to  $O_\sigma$  near any point.

(ii) Since  $0 \notin \tau$ , we have  $X^{-P}G_\tau = X^{-P}f_\tau$ . As  $X^{-P}f_\tau$  does not involve the variable  $T$ ,  $Y \cap O_\sigma \rightarrow \mathbf{A}_k^1$  can be obtained from  $Z \rightarrow \text{Spec } k$  by base change.

Suppose  $f$  is non-degenerate. Let's prove  $Z$  is smooth. Since  $\mathbf{Z}^n/\mathbf{Z}^n \cap \sigma^\perp$  is torsion free,  $\mathbf{Z}^n \cap \sigma^\perp$  is a direct factor of  $\mathbf{Z}^n$ . Choose a basis  $\{e_1, \dots, e_n\}$  of  $\mathbf{Z}^n$  so that  $\{e_1, \dots, e_r\}$  is a basis of  $\mathbf{Z}^n \cap \sigma^\perp$ . Let  $(Y_1, \dots, Y_n)$  be the coordinates with respect to this basis. Then  $X^{-P}f_\tau \in k[\mathbf{Z}^n \cap \sigma^\perp]$  only depends on the variables  $Y_1, \dots, Y_r$ . Since  $f$  is non-degenerate with respect to  $\Delta_\infty(f)$  and  $0 \notin \tau$ , the locus of  $X^{-P}f_\tau = 0$  in  $\mathbf{T}_k^n$  is smooth over  $k$ . As  $X^{-P}f_\tau$  depends only on  $Y_1, \dots, Y_r$ , the locus of  $X^{-P}f_\tau = 0$  in  $\mathbf{T}_k^r = \text{Spec } k[\mathbf{Z}^n \cap \sigma^\perp]$  is smooth over  $k$ , that is,  $Z$  is smooth over  $k$ .

(iii) Now suppose  $0 \in \tau$ . We can then take  $P = 0$ . We have  $G_\tau = f_\tau - T$ . Since we have an isomorphism

$$A[\mathbf{Z}^n \cap \sigma^\perp]/(f_\tau - T) \cong k[\mathbf{Z}^n \cap \sigma^\perp], \quad T \mapsto f_\tau,$$

the locus of  $G_\tau = 0$  in  $O_\sigma = \text{Spec } A[\mathbf{Z}^n \cap \sigma^\perp]$  can be identified with  $O_{\sigma k} = \text{Spec } k[\mathbf{Z}^n \cap \sigma^\perp]$ , and  $g|_{Y \cap O_\sigma}$  can be identified with the morphism  $f_\tau : O_{\sigma k} \rightarrow \mathbf{A}_k^1$ .

Again choose a basis  $\{e_1, \dots, e_n\}$  of  $\mathbf{Z}^n$  so that  $\{e_1, \dots, e_r\}$  is a basis of  $\mathbf{Z}^n \cap \sigma^\perp$ , where  $r = \dim(\sigma^\perp)$ . Let  $(Y_1, \dots, Y_n)$  be the coordinates with respect to this basis. Then  $f_\tau$  depend only on the coordinates  $Y_1, \dots, Y_r$ . Note that

$$\Delta_\infty(f_\tau(Y_1, \dots, Y_r)) = \tau.$$

If  $f$  is non-degenerate with respect to  $\Delta_\infty(f)$ , then  $f_\tau(Y_1, \dots, Y_r)$  is non-degenerate with respect to  $\Delta_\infty(f_\tau(Y_1, \dots, Y_r))$ . Furthermore, if  $\sigma \in \Sigma(\Delta_\infty(f))$ , then by Proposition 1.2, we have  $\dim(\sigma) = n - \dim(\tau)$ , and hence

$$\dim(\Delta_\infty(f_\tau(Y_1, \dots, Y_r))) = \dim(\tau) = \dim(\sigma^\perp) = r.$$

By [DL] 3.5,  $f_\tau : \mathbf{T}_k^r \rightarrow \mathbf{A}_k^1$  is then smooth outside finitely many closed points. So  $Y \cap O_\sigma$  is smooth over  $\mathbf{A}_k^1$  outside finitely many closed points under our assumption.

**Lemma 3.3.** Keep the notations in Lemma 3.2. Suppose  $\dim(\Delta_\infty(f)) = n$ ,  $f$  is non-degenerate, and  $\Sigma$  is a subdivision of  $\Sigma(\Delta_\infty(f))$ . Let  $j : \mathbf{T}_k^n \rightarrow X_k(\Sigma)$  be the immersion of the open dense torus in  $X_k(\Sigma)$ , and let  $(j_{!*}(\mathcal{K}_\chi[n]))|_Y$  and  $(j_*\mathcal{K}_\chi)|_Y$  the inverse images under the composition

$$Y \rightarrow X_A(\Sigma) \rightarrow X_k(\Sigma)$$

of  $j_{!*}(\mathcal{K}_\chi[n])$  and  $j_*\mathcal{K}_\chi$ , respectively. Then we have

$$j_{Y!}(\mathcal{K}_\chi[n]) \cong (j_{!*}(\mathcal{K}_\chi[n]))|_Y,$$

$$j_{Y*}\mathcal{K}_\chi \cong (j_*\mathcal{K}_\chi)|_Y.$$

**Proof.** By Lemma 3.2,  $Y \cap O_\sigma$  is smooth over  $k$  for any  $\sigma \in \Sigma$ . Our assertion then follows from Proposition 2.7.

**Proposition 3.4.** Keep the notations in Lemma 3.2. Suppose  $\dim(\Delta_\infty(f)) = n$ ,  $f$  is non-degenerate, and  $\Sigma$  is a subdivision of  $\Sigma(\Delta_\infty(f))$  with the property that for any  $\sigma \in \Sigma$  with  $0 \in F_{\Delta_\infty(f)}(\sigma)$ , we have  $\sigma \in \Sigma(\Delta_\infty(f))$ .

(i) Outside finitely many closed points in  $Y$ ,  $g : Y \rightarrow \mathbf{A}_k^1$  is universally locally acyclic relative to  $j_{Y!}(\mathcal{K}_\chi[n])$ , and relative to  $j_{Y*}\mathcal{K}_\chi$ .

(ii) For any  $\sigma \in \Sigma$ ,  $g|_{Y \cap V(\sigma)} \rightarrow \mathbf{A}_k^1$  is universally locally acyclic relative to  $(j_{Y*}\mathcal{K}_\chi)|_{Y \cap V(\sigma)}$  outside finitely many closed points.

**Proof.**

(i) By Lemma 3.3, we have

$$\begin{aligned} j_{Y!}(\mathcal{K}_\chi[n]) &\cong (j_!(\mathcal{K}_\chi[n]))|_Y, \\ j_{Y*}\mathcal{K}_\chi &\cong (j_*\mathcal{K}_\chi)|_Y, \end{aligned}$$

By Lemma 3.2,  $Y \cap O_\sigma$  is smooth over  $A$  outside finitely many closed points for any  $\sigma \in \Sigma$ . So by Proposition 2.5,  $Y$  is universally locally acyclic over  $A$  relative to  $(j_!(\mathcal{K}_\chi[n]))|_Y$  and relative to  $(j_*\mathcal{K}_\chi)|_Y$  outside finitely many closed points. Thus  $g : Y \rightarrow \text{Spec } A = \mathbf{A}_k^1$  is universally locally acyclic relative  $j_{Y!}(\mathcal{K}_\chi[n])$  and relative to  $j_{Y*}\mathcal{K}_\chi$  outside finitely many closed points.

(ii) Let

$$p_1 : \mathbf{T}_k^n = \text{Spec } k[\mathbf{Z}^n] \rightarrow O_{\sigma_k} = \text{Spec } k[\mathbf{Z}^n \cap \sigma^\perp]$$

be the projection. If  $\mathcal{K}_\chi$  is not the inverse image of a Kummer sheaf on  $O_{\sigma_k}$ , then by Proposition 2.4,  $j_*\mathcal{K}_\chi$  vanishes on  $V(\sigma)_k = \text{Spec } k[\mathbf{Z}^n \cap \sigma]$ . As  $j_{Y*}\mathcal{K}_\chi = (j_*\mathcal{K}_\chi)|_Y$ ,  $j_{Y*}\mathcal{K}_\chi$  vanishes on  $Y \cap V(\sigma)$ . Hence  $g|_{Y \cap V(\sigma)} \rightarrow \mathbf{A}_k^1$  is universally locally acyclic relative to  $(j_{Y*}\mathcal{K}_\chi)|_{Y \cap V(\sigma)}$  in this case.

Now suppose  $\mathcal{K}_\chi = p_1^*\mathcal{K}_{\chi_1}$  for some Kummer sheaf  $\mathcal{K}_{\chi_1}$  on  $O_{\sigma_k}$ . Let  $\Sigma_1 = \text{star}(\sigma)$  and let  $j_1 : O_{\sigma_k} \rightarrow X_k(\Sigma_1) = V(\sigma)_k$  be the immersion of the open dense torus. By Proposition 2.4, we have

$$(j_*\mathcal{K}_\chi)|_{V(\sigma)_k} \cong j_{1*}\mathcal{K}_{\chi_1}.$$



So we have

$$\begin{aligned} (j_{Y*}\mathcal{K}_\chi)|_{Y \cap V(\sigma)} &\cong (j_*\mathcal{K}_\chi)|_{Y \cap V(\sigma)} \\ &\cong (j_{1*}\mathcal{K}_{\chi_1})|_{Y \cap V(\sigma)}. \end{aligned}$$

For each  $\tau \in \Sigma_1$ , by Lemma 3.2,  $Y \cap O_\tau$  is smooth over  $A$  outside finitely many points. So by Proposition 2.5 applied to the toric scheme  $X_A(\Sigma_1) = V(\sigma)$  and the Cartier divisor  $Y \cap V(\sigma)$ ,  $Y \cap V(\sigma)$  is universally locally acyclic over  $A$  relative to  $(j_{1*}\mathcal{K}_{\chi_1})|_{Y \cap V(\sigma)}$  outside finitely many closed points. Hence  $Y \cap V(\sigma)$  is universally locally acyclic over  $A$  relative to  $(j_{Y*}\mathcal{K}_\chi)|_{Y \cap V(\sigma)}$  outside finitely many closed points.

**Lemma 3.5.** Let  $Y$  be a scheme of finite type over a field  $k$ ,  $\mathcal{F}$  a  $\overline{\mathbf{Q}}_l$ -sheaf on  $Y$ ,  $J$  a finite set, and  $Y_j$  a closed subscheme of  $Y$  for each  $j \in J$ . For any  $I \subset J$ , let

$$Y_I = \bigcap_{j \in I} Y_j, \quad Y_I^\circ = Y_I - \bigcup_{j \in J-I} Y_j.$$

We have  $Y_\emptyset = Y$ , and we denote  $Y_\emptyset^\circ = Y - \bigcup_{j \in J} Y_j$  by  $Y^\circ$ .

(i) Let  $g : Y \rightarrow \mathbf{A}_k^1$  be a  $k$ -morphism. Suppose  $R^i(g|_{Y_I^\circ})_!\mathcal{F}$  are tame at  $\infty$  for all  $I \subset J$  and all  $i$ . Then  $R^i(g|_{Y_I})_!\mathcal{F}$  are tame at  $\infty$  for all  $I \subset J$  and all  $i$ . In particular,  $R^i g_!\mathcal{F}$  are tame at  $\infty$  for all  $i$ .

(ii) Conversely, suppose  $R^i(g|_{Y_I})_!\mathcal{F}$  are tame at  $\infty$  for all  $I \subset J$  and all  $i$ . Then  $R^i(g|_{Y_I^\circ})_!\mathcal{F}$  are tame at  $\infty$  for all  $I \subset J$  and all  $i$ . In particular,  $R^i(g|_{Y^\circ})_!\mathcal{F}$  are tame at  $\infty$  for all  $i$ .

(iii) Let  $\phi$  be an integer valued function on the power set of  $J$  with the property  $\phi(I') \leq \phi(I)$  for any  $I \subset I' \subset J$ . Suppose  $H_c^i(Y_I^\circ \otimes_k \overline{k}, \mathcal{F}) = 0$  for all  $I \subset J$  and all  $i > \phi(I)$ . Then  $H_c^i(Y_I \otimes_k \overline{k}, \mathcal{F}) = 0$  for all  $I \subset J$  and all  $i > \phi(I)$ . In particular,  $H_c^i(Y \otimes_k \overline{k}, \mathcal{F}) = 0$  for all  $i > \phi(\emptyset)$ .

(iv) Let  $\phi$  be as in (iii). Suppose furthermore that for any  $I \subset I' \subset J$  with  $Y_{I'} \neq \emptyset$ ,  $Y_I$ , we have  $\phi(I') < \phi(I)$ . If  $H_c^i(Y_I \otimes_k \overline{k}, \mathcal{F}) = 0$  for all  $I \subset J$  and all  $i > \phi(I)$ , then  $H_c^i(Y_I^\circ \otimes_k \overline{k}, \mathcal{F}) = 0$  for all  $I \subset J$  and all  $i > \phi(I)$ . In particular,  $H_c^i(Y^\circ \otimes_k \overline{k}, \mathcal{F}) = 0$  for all  $i > \phi(\emptyset)$ .

**Proof.** We will prove (i) and (iv). The proof of (ii) and (iii) is similar.

(i) It suffices to prove  $R^i g_!\mathcal{F}$  are tame at  $\infty$  for all  $i$ . Indeed, applying this result to  $g|_{Y_I}$  and the closed subschemes  $Y_I \cap Y_j$  ( $j \in J - I$ ), we see  $R^i(g|_{Y_I})_!\mathcal{F}$  are tame at  $\infty$  for all  $i$  and all  $I \subset J$ . We use induction on the number of elements of  $J$ . The case where  $J = \emptyset$  is trivial. If  $J = \{1\}$ , we

have  $Y_\emptyset^\circ = Y - Y_1$  and  $Y_J^\circ = Y_1$ . By our assumption,  $R^i(g|_{Y-Y_1})_!\mathcal{F}$  and  $R^i(g|_{Y_1})_!\mathcal{F}$  are tame at  $\infty$ . We have a long exact sequence

$$\cdots \rightarrow R^i(g|_{Y-Y_1})_!\mathcal{F} \rightarrow R^i g_!\mathcal{F} \rightarrow R^i(g|_{Y_1})_!\mathcal{F} \rightarrow \cdots.$$

It follows that  $R^i g_!\mathcal{F}$  are tame at  $\infty$ . Suppose  $J = \{1, \dots, m\}$  ( $m \geq 2$ ) and suppose our assertion holds for those  $J$  with less than  $m$  elements. Applying the induction hypothesis to  $Y - Y_1$  and the closed subschemes  $Y_j - Y_1$  ( $j \in J - \{1\}$ ), we see  $R^i(g|_{Y-Y_1})_!\mathcal{F}$  are tame at  $\infty$ . Applying the induction hypothesis to  $Y_1$  and the closed subschemes  $Y_1 \cap Y_j$  ( $j \in J - \{1\}$ ), we see  $R^i(g|_{Y_1})_!\mathcal{F}$  are tame at  $\infty$ . It follows from the previous long exact sequence that  $R^i g_!\mathcal{F}$  are tame at  $\infty$ .

(iv) As in (i), it suffices to show  $H_c^i(Y^\circ \otimes_k \bar{k}, \mathcal{F}) = 0$  for  $i > \phi(\emptyset)$ . We use induction on the number of element of  $J$ . The case where  $J = \emptyset$  is trivial. If  $J = \{1\}$ , we have  $Y_\emptyset = Y$ ,  $Y_J = Y_1$  and  $Y^\circ = Y - Y_1$ . The cases where  $Y_J = \emptyset$  or  $Y_J = Y_\emptyset$  are trivial. Suppose  $Y_J \neq \emptyset, Y_\emptyset$ . Then  $\phi(J) < \phi(\emptyset)$ . We have a long exact sequence

$$\cdots \rightarrow H_c^{i-1}(Y_1 \otimes_k \bar{k}, \mathcal{F}) \rightarrow H_c^i((Y - Y_1) \otimes_k \bar{k}, \mathcal{F}) \rightarrow H_c^i(Y \otimes_k \bar{k}, \mathcal{F}) \rightarrow \cdots.$$

By our assumption, we have  $H_c^{i-1}(Y_1 \otimes_k \bar{k}, \mathcal{F}) = 0$  for  $i - 1 > \phi(J)$  and  $H_c^i(Y \otimes_k \bar{k}, \mathcal{F}) = 0$  for  $i > \phi(\emptyset)$ . It follows that  $H_c^i((Y - Y_1) \otimes_k \bar{k}, \mathcal{F}) = 0$  for  $i > \phi(\emptyset)$ . Let  $J = \{1, \dots, m\}$  ( $m \geq 2$ ) and suppose our assertion holds for those  $J$  with less than  $m$  elements. Applying the induction hypothesis to  $Y$  and the closed subschemes  $Y_j$  ( $j \in J - \{1\}$ ), we see  $H^i((Y - \bigcup_{j \neq 1} Y_j) \otimes_k \bar{k}, \mathcal{F}) = 0$  for all  $i \geq \phi(\emptyset)$ . Applying the induction hypothesis to  $Y_1$  and the closed subschemes  $Y_1 \cap Y_j$  ( $j \in J - \{1\}$ ), we see  $H^{i-1}((Y_1 - \bigcup_{j \neq 1} Y_j) \otimes_k \bar{k}, \mathcal{F}) = 0$  for all  $i - 1 \geq \phi(\{1\})$ . We have a long exact sequence

$$\cdots \rightarrow H^{i-1}((Y_1 - \bigcup_{j \neq 1} Y_j) \otimes_k \bar{k}, \mathcal{F}) \rightarrow H^i((Y - \bigcup_j Y_j) \otimes_k \bar{k}, \mathcal{F}) \rightarrow H^i((Y - \bigcup_{j \neq 1} Y_j) \otimes_k \bar{k}, \mathcal{F}) \rightarrow \cdots.$$

If  $Y_1 \neq \emptyset, Y$ , then  $\phi(\{1\}) < \phi(\emptyset)$ . It follows that  $H^i((Y - \bigcup_j Y_j) \otimes_k \bar{k}, \mathcal{F}) = 0$  for all  $i > \phi(\emptyset)$ . If  $Y_1 = \emptyset$ , then  $H^i((Y - \bigcup_j Y_j) \otimes_k \bar{k}, \mathcal{F}) = H^i((Y - \bigcup_{j \neq 1} Y_j) \otimes_k \bar{k}, \mathcal{F}) = 0$  for  $i \geq \phi(\emptyset)$ . If  $Y_1 = Y$ , then  $Y - \bigcup_j Y_j = \emptyset$  and  $H^i((Y - \bigcup_j Y_j) \otimes_k \bar{k}, \mathcal{F}) = 0$  for all  $i$ .

**Lemma 3.6.** Let  $k$  be a field. Suppose  $f : \mathbf{T}_k^n \rightarrow \mathbf{A}_k^1$  is a  $k$ -morphism defined by a Laurent polynomial  $f \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  that is non-degenerate with respect to  $\Delta_\infty(f)$ . Then  $R^i f_! \mathcal{K}_\chi$  are tame at  $\infty$  for all  $i$ .

**Proof.** For the Kummer covering

$$[m] : \mathbf{T}_k^n \rightarrow \mathbf{T}_k^n, x \mapsto x^m,$$

we have

$$[m]_* \overline{\mathbf{Q}}_l \cong \bigoplus_{\chi} \mathcal{K}_{\chi},$$

where  $\chi : \mu_m(k)^n \rightarrow \overline{\mathbf{Q}}_l^*$  goes over the set of characters of  $\mu_m(k)^n$ . The composition  $f \circ [m]$  is defined by the Laurent polynomial  $f(x_1^m, \dots, x_n^m)$ . It is not hard to see that this Laurent polynomial is non-degenerate with respect to its Newton polyhedron at  $\infty$ . So by [DL] 4.2,  $R^i(f \circ [m])_* \overline{\mathbf{Q}}_l$  are tame at  $\infty$  for all  $i$ . We have

$$\begin{aligned} R^i(f \circ [m])_* \overline{\mathbf{Q}}_l &\cong R^i f_*([m]_* \overline{\mathbf{Q}}_l) \\ &\cong \bigoplus_{\chi} R^i f_* \mathcal{K}_{\chi}. \end{aligned}$$

So  $R^i f_* \mathcal{K}_{\chi}$  are tame at  $\infty$  for all  $i$  and all  $\chi$ .

**Lemma 3.7.** Keep the notations in Lemma 3.2. Suppose  $f$  is non-degenerate,  $\dim(\Delta_{\infty}(f)) = n$ ,  $\Sigma$  is a subdivision of  $\Sigma(\Delta_{\infty}(f))$ , and  $\sigma \in \Sigma$  so that  $0 \notin F_{\Delta_{\infty}(f)}(\sigma)$ . Then  $R^i(g|_{Y \cap O_{\sigma}})_*(j_{Y!}(\mathcal{K}_{\chi}[n]))$  and  $R^i(g|_{Y \cap O_{\sigma}})_*(j_{Y*} \mathcal{K}_{\chi})$  are constant sheaves.

**Proof.** By Lemma 3.2 (ii), we have a commutative diagram of Cartesian squares

$$\begin{array}{ccc} Y \cap O_{\sigma} & \rightarrow & Z \\ \downarrow & & \downarrow \\ X_A(\Sigma) & \rightarrow & X_k(\Sigma) \\ \downarrow & & \downarrow \\ \mathbf{A}_k^1 & \rightarrow & \text{Spec } k, \end{array}$$

where  $Z$  is a closed subscheme of  $O_{\sigma_k}$  and we regard  $O_{\sigma_k}$  as a subscheme of  $X_k(\Sigma)$ . Let  $j : \mathbf{T}_k^n \rightarrow X_k(\Sigma)$  be the immersion of the open dense torus in  $X_k(\Sigma)$ . By Lemma 3.3, we have

$$\begin{aligned} j_{Y!}(\mathcal{K}_{\chi}[n]) &\cong (j_!(\mathcal{K}_{\chi}[n]))|_Y, \\ j_{Y*}(\mathcal{K}_{\chi}) &\cong (j_* \mathcal{K}_{\chi})|_Y. \end{aligned}$$

Fix notations by the following diagram

$$\begin{array}{ccc} Y \cap O_{\sigma} & \xrightarrow{\rho'} & Z \\ g \downarrow & & \downarrow \lambda \\ \mathbf{A}_k^1 & \xrightarrow{\rho} & \text{Spec } k, \end{array}$$

By the proper base change theorem, we have

$$\begin{aligned}
R^i(g|_{Y \cap O_\sigma})!(j_{Y*}\mathcal{K}_\chi) &\cong R^i(g|_{Y \cap O_\sigma})!((j_*\mathcal{K}_\chi)|_{Y \cap O_\sigma}) \\
&\cong R^i(g|_{Y \cap O_\sigma})!\rho'^*((j_*\mathcal{K}_\chi)|_Z) \\
&\cong \rho^*R^i\lambda_!((j_*\mathcal{K}_\chi)|_Z),
\end{aligned}$$

that is,

$$R^i(g|_{Y \cap O_\sigma})!(j_{Y*}\mathcal{K}_\chi) \cong \rho^*R^i\lambda_!((j_*\mathcal{K}_\chi)|_Z).$$

Hence  $R^i(g|_{Y \cap O_\sigma})!(j_{Y*}\mathcal{K}_\chi)$  are constant for all  $i$ . Similarly,  $R^i(g|_{Y \cap O_\sigma})!(j_{Y!*}(\mathcal{K}_\chi[n]))$  are also constant for all  $i$ .

**Proposition 3.8.** Keep the notations in Lemma 3.2. Suppose  $\dim(\Delta_\infty(f)) = n$ ,  $f$  is non-degenerate, and  $\Sigma$  is a subdivision of  $\Sigma(\Delta_\infty(f))$ . For any  $\sigma \in \Sigma$ ,  $R^i(g|_{Y \cap V(\sigma)})_*(j_{Y!*}(\mathcal{K}_\chi[n]))$  and  $R^i(g|_{Y \cap V(\sigma)})_*(j_{Y*}\mathcal{K}_\chi)$  are tame at  $\infty$  for all  $i$ . In particular, taking  $\sigma = 0$ , we see  $R^i g_*(j_{Y!*}(\mathcal{K}_\chi[n]))$  and  $R^i g_*(j_{Y*}\mathcal{K}_\chi)$  are tame at  $\infty$  for all  $i$ .

**Proof.** Let  $J$  be the set of one dimensional cones in  $\Sigma$ , and for each  $\sigma \in J$ , let  $Y_\sigma = Y \cap V(\sigma)$ .

For any subset  $I = \{\sigma_1, \dots, \sigma_r\}$  of  $J$ , we have

$$\begin{aligned}
Y_I &= \bigcap_{i=1}^r Y_{\sigma_i} \\
&= Y \cap \left( \bigcap_{i=1}^r V(\sigma_i) \right) \\
&= Y \cap \left( \bigcap_{i=1}^r \coprod_{\sigma_i \prec \gamma} O_\gamma \right) \\
&= Y \cap \left( \coprod_{\sigma_1, \dots, \sigma_r \prec \gamma} O_\gamma \right).
\end{aligned}$$

So if  $\sigma$  is the smallest cone in  $\Sigma$  containing  $\sigma_1, \dots, \sigma_r$ , then we have

$$Y_I = Y \cap V(\sigma).$$

If there is no cone in  $\Sigma$  containing all  $\sigma_1, \dots, \sigma_r$ , then  $Y_I = \emptyset$ . Note that since  $\sigma_1, \dots, \sigma_r$  are one-dimensional cones in  $\Sigma$ , the smallest cone in  $\Sigma$  containing  $\sigma_1, \dots, \sigma_r$  is exactly the cone generated by  $\sigma_1, \dots, \sigma_r$  if this cone lies in  $\Sigma$ . Suppose  $J - I = \{\sigma_{r+1}, \dots, \sigma_s\}$ . Then we have

$$Y_I^\circ = Y_I - \bigcup_{i=r+1}^s Y_{\sigma_i}$$

$$\begin{aligned}
&= Y \cap \left( \coprod_{\sigma_1, \dots, \sigma_r \prec \gamma} O_\gamma - \bigcup_{i=r+1}^s \coprod_{\sigma_j \prec \gamma} O_\gamma \right) \\
&= Y \cap \left( \coprod_{\substack{\sigma_1, \dots, \sigma_r \prec \gamma \\ \sigma_{r+1}, \dots, \sigma_s \not\prec \gamma}} O_\gamma \right) \\
&= Y \cap O_\sigma,
\end{aligned}$$

where  $\sigma$  is again the cone in  $\Sigma$  generated by  $\sigma_1, \dots, \sigma_r$ . (If the cone generated by  $\sigma_1, \dots, \sigma_r$  does not lie in  $\Sigma$ , then  $Y_I^\circ = \emptyset$ .) By Lemma 3.5 (i), to prove our assertion, it suffices to show that for any  $\sigma \in \Sigma$ ,  $R^i(g|_{Y \cap O_\sigma})_*(j_{Y!}(\mathcal{K}_\chi[n]))$  and  $R^i(g|_{Y \cap O_\sigma})_*(j_{Y*}\mathcal{K}_\chi)$  are tame at  $\infty$  for all  $i$ . If  $0 \notin F_{\Delta_\infty(f)}(\sigma)$ , this follows from Lemma 3.7. Suppose  $0 \in F_{\Delta_\infty(f)}(\sigma)$ . By Lemma 3.2 (iii),  $Y \cap O_\sigma$  can be identified with  $O_{\sigma_k}$  and  $g|_{Y \cap O_\sigma} : Y \cap O_\sigma \rightarrow \mathbf{A}_k^1$  can be identified with  $f_\tau : O_{\sigma_k} \rightarrow \mathbf{A}_k^1$ , where  $\tau = F_{\Delta_\infty(f)}(\sigma)$ . Using Lemma 3.3, one can check  $(j_{Y!}(\mathcal{K}_\chi[n]))|_{Y \cap O_\sigma}$  and  $(j_{Y!}\mathcal{K}_\chi)|_{Y \cap O_\sigma}$  are identified with  $(j_{!*}(\mathcal{K}_\chi[n]))|_{O_{\sigma_k}}$  and  $(j_*\mathcal{K}_\chi)|_{O_{\sigma_k}}$ , respectively. So it suffice to verify  $R^i f_{\tau!}((j_{!*}(\mathcal{K}_\chi[n]))|_{O_{\sigma_k}})$  and  $R^i f_{\tau!}((j_*\mathcal{K}_\chi)|_{O_{\sigma_k}})$  are tame at  $\infty$  for all  $i$ . This follows from Lemma 3.6 and the description of  $(j_{!*}(\mathcal{K}_\chi[n]))|_{O_{\sigma_k}}$  and  $(j_*\mathcal{K}_\chi)|_{O_{\sigma_k}}$  in Lemma 2.3. This finishes the proof of the proposition.

A Laurent polynomial

$$G(X_1, \dots, X_n) = \sum_{i \in \mathbf{Z}^n} a_i X^i \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$$

is called *0-non-degenerate* with respect to its Newton polyhedron  $\Delta(G)$  if for any face  $\tau$  of  $\Delta(G)$ , the locus of  $G_\tau = \sum_{i \in \tau} a_i X^i = 0$  in  $\mathbf{T}_k^n$  is smooth over  $k$ .

**Proposition 3.9.** Let  $G(X_1, \dots, X_n) \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  be a Laurent polynomial that is 0-non-degenerate with respect to  $\Delta(G)$ . Then

$$\chi_c(\mathbf{T}_k^n \cap G^{-1}(0), \mathcal{K}_\chi) = (-1)^{n-1} n! \text{vol}(\Delta(G)),$$

where

$$\chi_c(\mathbf{T}_k^n \cap G^{-1}(0), \mathcal{K}_\chi) = \sum_i (-1)^i \dim H_c^i(\mathbf{T}_k^n \cap G^{-1}(0), \mathcal{K}_\chi)$$

is the Euler characteristic.

**Proof.** Let  $r = \dim(\Delta(G))$ . If  $r < n$ , then after a suitable change of coordinates on the torus, we may assume  $G$  depends on  $r$  variables. Without loss of generality, assume

$$G(X_1, \dots, X_n) = H(X_1, \dots, X_r)$$

for some Laurent polynomial  $H(X_1, \dots, X_r)$ . Then we have

$$\mathbf{T}_k^n \cap G^{-1}(0) = (\mathbf{T}_k^r \cap H^{-1}(0)) \times_k \mathbf{T}_k^{n-r}.$$

Let

$$p_1 : \mathbf{T}_k^n = \mathbf{T}_k^r \times \mathbf{T}_k^{n-r} \rightarrow \mathbf{T}_k^r, \quad p_2 : \mathbf{T}_k^n = \mathbf{T}_k^r \times \mathbf{T}_k^{n-r} \rightarrow \mathbf{T}_k^{n-r}$$

be the projections. We can find Kummer sheaves  $\mathcal{K}_{\chi_1}$  on  $\mathbf{T}_k^r$  and  $\mathcal{K}_{\chi_2}$  on  $\mathbf{T}_k^{n-r}$  so that

$$\mathcal{K}_\chi \cong p_1^* \mathcal{K}_{\chi_1} \otimes p_2^* \mathcal{K}_{\chi_2}.$$

By the Künneth formula, we have

$$\begin{aligned} \chi_c(\mathbf{T}_k^n \cap G^{-1}(0), \mathcal{K}_\chi) &= \chi_c((\mathbf{T}_k^r \cap H^{-1}(0)) \times_k \mathbf{T}_k^{n-r}, p_1^* \mathcal{K}_{\chi_1} \otimes p_2^* \mathcal{K}_{\chi_2}) \\ &= \chi_c(\mathbf{T}_k^r \cap H^{-1}(0), \mathcal{K}_{\chi_1}) \chi_c(\mathbf{T}_k^{n-r}, \mathcal{K}_{\chi_2}). \end{aligned}$$

We have

$$\chi_c(\mathbf{T}_k^{n-r}, \mathcal{K}_{\chi_2}) = 0.$$

(To see this, we use the Künneth formula to reduce to the case where the dimension of the torus is 1. We then use the Grothendieck-Ogg-Shafarevich formula.) It follows that

$$\chi_c(\mathbf{T}_k^n \cap G^{-1}(0), \mathcal{K}_\chi) = 0 = (-1)^{n-1} n! \text{vol}(\Delta(G)).$$

Now suppose  $\dim \Delta(G) = n$ . Let  $\Sigma$  be a regular fan that is a subdivision of  $\Sigma(\Delta(G))$ . Let  $Y$  be the scheme theoretic closure in  $X_k(\Sigma)$  of the locus  $G = 0$  in  $\mathbf{T}_k^n$ . By Proposition 3.1 and the assumption that  $G$  is 0-non-degenerate with respect to  $\Delta(G)$ ,  $Y \cap O_\sigma$  is smooth over  $k$  for any  $\sigma \in \Sigma$ . By [DL] 2.3,  $Y$  is smooth over  $k$ . So  $Y$  is a smooth compactification of the locus  $G = 0$  in  $\mathbf{T}_k^n$ . Note that  $(\mathcal{K}_\chi)|_{\mathbf{T}_k^n \cap G^{-1}(0)}$  is tamely ramified along  $Y - (\mathbf{T}_k^n \cap G^{-1}(0))$  in the sense of [I1] 2.6. (Indeed, the inverse image of  $\mathcal{K}_\chi$  under the Kummer covering

$$[m] : \mathbf{T}_k^n \rightarrow \mathbf{T}_k^n, \quad x \mapsto x^m$$

is constant.) So by [I1] 2.7, we have

$$\chi_c(\mathbf{T}_k^n \cap G^{-1}(0), \mathcal{K}_\chi) = \chi_c(\mathbf{T}_k^n \cap G^{-1}(0), \overline{\mathbf{Q}}_l).$$

(To apply [I1] 2.7, we only require  $Y$  is a normal compactification of the locus  $G = 0$  in  $\mathbf{T}_k^n$ . If we take  $\Sigma$  to be just an arbitrary subdivision of  $\Sigma(\Delta(G))$ , then using [DL] 2.3, one can verify  $Y$  is normal. So  $\Sigma$  being regular is not necessary for our purpose.) By [DL] 2.7, we have

$$\chi_c(\mathbf{T}_k^n \cap G^{-1}(0), \overline{\mathbf{Q}}_l) = (-1)^{n-1} n! \text{vol}(\Delta(G)).$$

So we have

$$\chi_c(\mathbf{T}_k^n \cap G^{-1}(0), \mathcal{K}_\chi) = (-1)^{n-1} n! \text{vol}(\Delta(G)).$$

#### 4. Cohomology and weights

In this section,  $k$  is a finite field of characteristic  $p$  with  $q$  elements,  $\psi : (k, +) \rightarrow \overline{\mathbf{Q}}_l^*$  a nontrivial additive character, and  $\mathcal{L}_\psi$  the lisse sheaf of rank 1 on  $\mathbf{A}_k^1$  obtained by pushing-forward the  $\mathbf{A}_k^1(k)$ -torsor

$$0 \rightarrow \mathbf{A}_k^1(k) \rightarrow \mathbf{A}_k^1 \xrightarrow{\mathcal{P}} \mathbf{A}_k^1 \rightarrow 0$$

by  $\psi^{-1}$ , where

$$\mathcal{P} : \mathbf{A}_k^1 \rightarrow \mathbf{A}_k^1, x \mapsto x^q - x$$

is the Artin-Schreier covering.

The following lemma is essentially Propositions 3.1 and 7.1 in [DL]. We include its proof for completeness.

**Lemma 4.1.** Let  $Y$  be a scheme of finite type over  $k$ , let  $g : Y \rightarrow \mathbf{A}_k^1$  a proper  $k$ -morphism, and let  $\mathcal{K}$  be an object in the derived category  $D_c^b(Y, \overline{\mathbf{Q}}_l)$  of  $\overline{\mathbf{Q}}_l$ -sheaves on  $Y$  defined in [D] 1.1.2. Suppose  $R^i g_* \mathcal{K}$  are tame at  $\infty$  for all  $i$ , and suppose  $g$  is locally acyclic relative to  $\mathcal{K}$  outside finitely many closed points.

(i) The canonical homomorphisms

$$H_c^i(Y \otimes_k \bar{k}, \mathcal{K} \otimes g^* \mathcal{L}_\psi) \rightarrow H^i(Y \otimes_k \bar{k}, \mathcal{K} \otimes g^* \mathcal{L}_\psi)$$

are isomorphisms for all  $i$ .

(ii) If  $\mathcal{K}$  is perverse, then

$$H_c^i(Y \otimes_k \bar{k}, \mathcal{K} \otimes g^* \mathcal{L}_\psi) = 0$$

for all  $i > 0$ .

(iii) Suppose  $Y$  is pure of dimension  $n$  and  $\mathcal{K}$  is a  $\overline{\mathbf{Q}}_l$ -sheaf. Then

$$H_c^i(Y \otimes_k \bar{k}, \mathcal{K} \otimes g^* \mathcal{L}_\psi) = 0$$

for all  $i > n$ .

**Proof.**

(i) Let  $\iota : \mathbf{A}_k^1 \hookrightarrow \mathbf{P}_k^1$  be the canonical open immersion. Since  $R^j g_* \mathcal{K}$  are tame at  $\infty$  and  $\mathcal{L}_\psi$  is totally wild at  $\infty$ , we have

$$(\iota_*(R^j g_* \mathcal{K} \otimes \mathcal{L}_\psi))_{\infty} = 0.$$

By [SGA 4 $\frac{1}{2}$ ] [Sommes trig.] Proposition 1.19 and Exemple 1.19.1, the canonical homomorphisms

$$H_c^i(\mathbf{A}_k^1, R^j g_* \mathcal{K} \otimes \mathcal{L}_\psi) \rightarrow H^i(\mathbf{A}_k^1, R^j g_* \mathcal{K} \otimes \mathcal{L}_\psi)$$

are isomorphisms. We have spectral sequences

$$\begin{aligned} E_2^{ij} &= H_c^i(\mathbf{A}_k^1, R^j g_* \mathcal{K} \otimes \mathcal{L}_\psi) \cong H_c^i(\mathbf{A}_k^1, R^j g_*(\mathcal{K} \otimes g^* \mathcal{L}_\psi)) \Rightarrow H_c^{i+j}(Y \otimes_k \bar{k}, \mathcal{K} \otimes g^* \mathcal{L}_\psi), \\ E_2^{ij} &= H^i(\mathbf{A}_k^1, R^j g_* \mathcal{K} \otimes \mathcal{L}_\psi) \cong H^i(\mathbf{A}_k^1, R^j g_*(\mathcal{K} \otimes g^* \mathcal{L}_\psi)) \Rightarrow H^{i+j}(Y \otimes_k \bar{k}, \mathcal{K} \otimes g^* \mathcal{L}_\psi), \end{aligned}$$

where to get the first spectral sequence, we use the assumption that  $g$  is proper. It follows that the canonical homomorphisms

$$H_c^i(Y \otimes_k \bar{k}, \mathcal{K} \otimes g^* \mathcal{L}_\psi) \rightarrow H^i(Y \otimes_k \bar{k}, \mathcal{K} \otimes g^* \mathcal{L}_\psi)$$

are isomorphisms.

(ii) Let  $s$  be an arbitrary closed point in  $\mathbf{A}_k^1$ ,  $S$  the henselization of  $\mathbf{A}_k^1$  at  $s$ , and  $\eta$  the generic point of  $S$ . Since  $g$  is proper, we have a long exact sequence

$$\begin{array}{ccccccc} \cdots & \rightarrow & H^j(X_{\bar{s}}, \mathcal{K}) & \rightarrow & H^j(X_{\bar{\eta}}, \mathcal{K}) & \rightarrow & H^j(X_{\bar{s}}, R\Phi(\mathcal{K})) \rightarrow \cdots, \\ & & \cong \uparrow & & \cong \uparrow & & \\ & & (R^j g_* \mathcal{K})_{\bar{s}} & & (R^j g_* \mathcal{K})_{\bar{\eta}} & & \end{array}$$



where  $R\Phi$  is the vanishing cycle functor. Since  $g$  is locally acyclic relative to  $\mathcal{K}$  outside finitely many closed points,  $R\Phi(\mathcal{K})$  is supported on finitely many closed points. By [I2] Corollaire 4.6,  $R\Phi(\mathcal{K})[-1]$  is perverse. It follows that

$$H^j(X_{\bar{s}}, R\Phi(\mathcal{K})) = \bigoplus_{x \in \text{supp}(R\Phi(\mathcal{K}))} (R^{j+1}\Phi(\mathcal{K})[-1])_{\bar{x}} = 0$$

for all  $j \geq 0$ . The above long exact sequence then shows that  $R^j g_*(\mathcal{K})$  are lisse on  $\mathbf{A}_k^1$  for all  $j \geq 1$ , and is an extension of a lisse sheaf by a punctual sheaf for  $j = 0$ . As  $R^j g_* \mathcal{K}$  are tame at  $\infty$ , all these lisse sheaves are constant on  $\mathbf{A}_k^1$ . It follows that  $H_c^i(\mathbf{A}_k^1, R^j g_* \mathcal{K} \otimes \mathcal{L}_\psi) = 0$  if  $j \geq 1$ , and if  $j = 0$  and  $i \geq 1$ . As in the proof of (i), the canonical homomorphisms

$$H_c^i(\mathbf{A}_{\bar{k}}^1, R^j g_* \mathcal{K} \otimes \mathcal{L}_\psi) \rightarrow H^i(\mathbf{A}_{\bar{k}}^1, R^j g_* \mathcal{K} \otimes \mathcal{L}_\psi)$$

are isomorphisms for all  $i$ . Combined with the Weak Lefschetz theorem, we get  $H_c^i(\mathbf{A}_{\bar{k}}^1, R^j g_* \mathcal{K} \otimes \mathcal{L}_\psi) = 0$  for all  $i \geq 2$ . It follows that  $H_c^i(\mathbf{A}_{\bar{k}}^1, R^j g_* \mathcal{K} \otimes \mathcal{L}_\psi) = 0$  for all  $i + j > 0$ . We have a spectral sequence

$$E_2^{ij} = H_c^i(\mathbf{A}_{\bar{k}}^1, R^j g_* \mathcal{K} \otimes \mathcal{L}_\psi) \Rightarrow H_c^{i+j}(Y \otimes_k \bar{k}, \mathcal{K} \otimes g^* \mathcal{L}_\psi).$$

So we have  $H_c^i(Y \otimes_k \bar{k}, \mathcal{K} \otimes g^* \mathcal{L}_\psi) = 0$  for all  $i > 0$ .

(iii) The proof is similar to that of (ii). Instead of using the perversity of  $R\Phi(\mathcal{K})[-1]$  in (i), we use the fact that  $R^j \Phi(\mathcal{K}) = 0$  for  $j > n - 1$  ([SGA 7] Exposé I, 4.2). We leave the details to the reader.

We are now ready to prove the main theorem of this paper.

**Theorem 4.2.** Let  $f : \mathbf{T}_k^n \rightarrow \mathbf{A}_k^1$  be a  $k$ -morphism defined by a Laurent polynomial  $f \in k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$  that is non-degenerate with respect to  $\Delta_\infty(f)$  and let  $\mathcal{K}_\chi$  be a Kummer sheaf on  $\mathbf{T}_k^n$ . Suppose  $\dim(\Delta_\infty(f)) = n$ . Then

- (i)  $H_c^i(\mathbf{T}_{\bar{k}}^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi) = 0$  for  $i \neq n$ .
- (ii)  $\dim(H_c^n(\mathbf{T}_{\bar{k}}^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)) = n! \text{vol}(\Delta_\infty(f))$ .
- (iii) Let

$$E(\mathbf{T}_k^n, f, \chi) = \sum_{w \in \mathbf{Z}} e_w T^w,$$

where  $e_w$  is the number of eigenvalues with weight  $w$  counted with multiplicities of the geometric Frobenius element  $F$  in  $\text{Gal}(\bar{k}/k)$  acting on  $H_c^i(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$ . Then  $E(\mathbf{T}_k^n, f, \chi)$  is a polynomial of degree  $\leq n$ , and

$$\begin{aligned} E(\mathbf{T}_k^n, f, \chi) &= E(\Delta_\infty(f), \chi), \\ e_n &= e(\Delta_\infty(f), \chi), \end{aligned}$$

where  $E(\Delta_\infty(f), \chi)$  and  $e(\Delta_\infty(f), \chi)$  are defined in the introduction.

(iv) If 0 is an interior point of  $\Delta_\infty(f)$ , then  $H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$  is pure of weight  $n$ .

**Proof.**

(i) Let  $\Sigma$  be a subdivision of  $\Sigma(\Delta_\infty(f))$  such that for any  $\sigma \in \Sigma$  with  $0 \in F_{\Delta_\infty(f)}(\sigma)$ , we have  $\sigma \in \Sigma(\Delta_\infty(f))$ . Let  $A = k[T]$ , let  $Y$  be the scheme theoretic closure in  $X_A(\Sigma)$  of the locus  $f - T = 0$  in  $\mathbf{T}_A^n$ , and let  $g : Y \rightarrow \mathbf{A}_k^1$  be the composition  $Y \rightarrow X_A(\Sigma) \rightarrow \text{Spec } A$ . Then  $g$  is proper and  $g|_{Y \cap \mathbf{T}_A^n}$  can be identified with  $f : \mathbf{T}_k^n \rightarrow \mathbf{A}_k^1$ . Let  $j_Y : \mathbf{T}_k^n \cong Y \cap \mathbf{T}_A^n \rightarrow Y$  be the open immersion induced by the immersion of the open dense torus  $\mathbf{T}_A^n$  in  $X_A(\Sigma)$ . By Proposition 3.4 (ii), for any  $\sigma \in \Sigma$ ,  $g : Y \cap V(\sigma) \rightarrow \mathbf{A}_k^1$  is locally acyclic relative to  $(j_{Y*} \mathcal{K}_\chi)|_{Y \cap V(\sigma)}$  outside finitely many closed points. By Proposition 3.8,  $R^i(g|_{Y \cap V(\sigma)})_*(j_{Y*} \mathcal{K}_\chi)$  are tame at  $\infty$  for all  $i$ . By Lemma 4.1 (iii), we have

$$H_c^i((Y \cap V(\sigma)) \otimes_k \bar{k}, j_{Y*} \mathcal{K}_\chi \otimes g^* \mathcal{L}_\psi) = 0$$

for all  $i > \dim(Y \cap V(\sigma)) = n - \dim(\sigma)$ . Let  $J$  be the set of nonzero cones in  $\Sigma$ . For each  $\sigma \in J$ , let  $Y_\sigma = Y \cap V(\sigma)$ . Then for any  $\sigma_1, \dots, \sigma_r \in J$ , we have

$$\begin{aligned} Y_{\sigma_1} \cap \dots \cap Y_{\sigma_r} &= Y \cap \left( \bigcap_{i=1}^r V(\sigma_i) \right) \\ &= Y \cap \left( \bigcap_{i=1}^r \coprod_{\sigma_i \prec \gamma} \mathcal{O}_\gamma \right) \\ &= Y \cap \left( \coprod_{\sigma_1, \dots, \sigma_r \prec \gamma} \mathcal{O}_\gamma \right). \end{aligned}$$

So if  $\sigma$  is the smallest cone in  $\Sigma$  containing  $\sigma_1, \dots, \sigma_r$ , then we have

$$Y_{\sigma_1} \cap \dots \cap Y_{\sigma_r} = Y \cap V(\sigma).$$

If there is no cone in  $\Sigma$  containing  $\sigma_1, \dots, \sigma_r$ , then  $Y_{\sigma_1} \cap \dots \cap Y_{\sigma_r}$  is empty. The condition of Lemma 3.5 (iv) holds with

$$\phi(\{\sigma_1, \dots, \sigma_r\}) = \dim(Y_{\sigma_1} \cap \dots \cap Y_{\sigma_r}).$$

So we have

$$H_c^i \left( \left( Y - \bigcup_{\sigma \in J} Y_\sigma \right) \otimes_k \bar{k}, j_{Y*} \mathcal{K}_\chi \otimes g^* \mathcal{L}_\psi \right) = 0$$

for all  $i > \dim(Y) = n$ , that is,

$$H_c^i(\mathbf{T}_{\bar{k}}^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi) = 0$$

for all  $i > n$ . By the Weak Lefschetz theorem and Poincaré duality, we have

$$H_c^i(\mathbf{T}_{\bar{k}}^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi) = 0$$

for all  $i < n$ . This proves (i).

(ii) By (i), we have

$$\dim(H_c^n(\mathbf{T}_{\bar{k}}^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)) = (-1)^n \chi_c(\mathbf{T}_{\bar{k}}^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi).$$

So it suffices to show

$$\chi_c(\mathbf{T}_{\bar{k}}^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi) = (-1)^n n! \text{vol}(\Delta_\infty(f)).$$

We have

$$\begin{aligned} \chi_c(\mathbf{T}_{\bar{k}}^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi) &= \chi_c(\mathbf{A}_{\bar{k}}^1, Rf_!(\mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)) \\ &= \chi_c(\mathbf{A}_{\bar{k}}^1, Rf_! \mathcal{K}_\chi \otimes \mathcal{L}_\psi). \end{aligned}$$

Since  $\mathcal{L}_\psi$  is a lisse sheaf of rank 1 on  $\mathbf{A}_{\bar{k}}^1$ , for any closed point  $x$  in  $\mathbf{A}_{\bar{k}}^1$ , we have

$$\text{sw}_x(Rf_! \mathcal{K}_\chi \otimes \mathcal{L}_\psi) = \text{sw}_x(Rf_! \mathcal{K}_\chi).$$

Applying the Grothendieck-Ogg-Shafarevich formula ([SGA 5] Exposé X, 7.1) to both  $\chi_c(\mathbf{A}_{\bar{k}}^1, Rf_! \mathcal{K}_\chi \otimes \mathcal{L}_\psi)$  and to  $\chi_c(\mathbf{A}_{\bar{k}}^1, Rf_! \mathcal{K}_\chi)$ , we see that

$$\chi_c(\mathbf{A}_{\bar{k}}^1, Rf_! \mathcal{K}_\chi \otimes \mathcal{L}_\psi) = \chi_c(\mathbf{A}_{\bar{k}}^1, Rf_! \mathcal{K}_\chi) + \text{sw}_\infty(Rf_! \mathcal{K}_\chi) - \text{sw}_\infty(Rf_! \mathcal{K}_\chi \otimes \mathcal{L}_\psi).$$

By Lemma 3.6,  $R^i f_! \mathcal{K}_\chi$  are tame at  $\infty$  for all  $i$ . So

$$\text{sw}_\infty(Rf_! \mathcal{K}_\chi) = 0.$$

Moreover,  $\mathcal{L}_\psi$  has Swan conductor 1 at  $\infty$  and has rank 1. So we have

$$\mathrm{sw}_\infty(Rf_!\mathcal{K}_\chi \otimes \mathcal{L}_\psi) = \mathrm{rank}(Rf_!\mathcal{K}_\chi) = \chi_c(f^{-1}(\bar{\eta}), \mathcal{K}_\chi),$$

where  $\eta$  is the generic point of  $\mathbf{A}_k^1$ . Therefore, we have

$$\chi_c(\mathbf{A}_k^1, Rf_!\mathcal{K}_\chi \otimes \mathcal{L}_\psi) = \chi_c(\mathbf{A}_k^1, Rf_!\mathcal{K}_\chi) - \chi_c(f^{-1}(\bar{\eta}), \mathcal{K}_\chi).$$

Note that

$$\chi_c(\mathbf{A}_k^1, Rf_!\mathcal{K}_\chi) = \chi_c(\mathbf{T}_k^n, \mathcal{K}_\chi) = 0.$$

So we have

$$\chi_c(\mathbf{A}_k^1, Rf_!\mathcal{K}_\chi \otimes \mathcal{L}_\psi) = -\chi_c(f^{-1}(\bar{\eta}), \mathcal{K}_\chi)$$

Therefore

$$\chi_c(\mathbf{A}_k^1, Rf_!\mathcal{K}_\chi \otimes \mathcal{L}_\psi) = -\chi_c(f^{-1}(a), \mathcal{K}_\chi)$$

for sufficiently general geometric point  $a$  in  $\mathbf{A}_k^1$ . We claim that for sufficiently general  $a$ , the Laurent polynomial  $G = f - a$  is 0-non-degenerate with respect to  $\Delta(G) = \Delta_\infty(f)$ . Hence by Proposition 3.9, we have

$$\begin{aligned} \chi_c(f^{-1}(a), \mathcal{K}_\chi) &= \chi_c(\mathbf{T}_k^n \cap G^{-1}(0), \mathcal{K}_\chi) \\ &= (-1)^{n-1} n! \mathrm{vol}(\Delta(G)) \\ &= (-1)^{n-1} n! \mathrm{vol}(\Delta_\infty(f)). \end{aligned}$$

Therefore

$$\chi_c(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi) = (-1)^n n! \mathrm{vol}(\Delta_\infty(f)).$$

Let's prove our claim. If  $\tau$  is a face of  $\Delta(G)$  that does not contain 0, then  $G_\tau = f_\tau$  (for any  $a$ ). Since  $f$  is non-degenerate with respect to  $\Delta_\infty(f)$ , the system of equations

$$\frac{\partial f_\tau}{\partial X_1} = \dots = \frac{\partial f_\tau}{\partial X_n} = 0$$

has no solution in  $\mathbf{T}_k^n$ . So the locus of  $G_\tau = 0$  in  $\mathbf{T}_k^n$  is smooth over  $k$  (for any  $a$ ). Now suppose  $\tau$  is a face of  $\Delta(G)$  containing 0. Then  $\Delta_\infty(f_\tau) = \tau$  and  $G_\tau = f_\tau - a$ . Note that  $f_\tau : \mathbf{T}_k^n \rightarrow \mathbf{A}_k^1$  is non-degenerate with respect to  $\Delta_\infty(f_\tau)$ . By [DL] 4.3, there exists a finite set  $S$  of closed points in  $\mathbf{A}_k^1$  such that  $f_\tau$  is smooth outside  $f_\tau^{-1}(S)$ . Choose  $a$  to be outside  $S$ . Then the locus of

$G_\tau = f_\tau - a = 0$  in  $\mathbf{T}_k^n$  is smooth over  $k$ . So for sufficiently general  $a$ ,  $G$  is 0-non-degenerate with respect to  $\Delta(G)$ . This finishes the proof of (ii).

(iii) By Proposition 3.4 (i),  $g : Y \rightarrow \mathbf{A}_k^1$  is locally acyclic relative to  $j_{Y!}(\mathcal{K}_\chi[n])$  outside finitely many closed points. By Proposition 3.8,  $R^i g_*(j_{Y!}(\mathcal{K}_\chi[n]))$  are tame at  $\infty$  for all  $i$ . By Lemma 4.1 (ii), we have

$$H_c^i(Y \otimes_k \bar{k}, j_{Y!}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi) = 0$$

for all  $i > 0$ . Replacing  $\chi$  by  $\chi^{-1}$  and  $\psi$  by  $\psi^{-1}$ , we see

$$H_c^i(Y \otimes_k \bar{k}, j_{Y!}(\mathcal{K}_{\chi^{-1}}[n]) \otimes g^* \mathcal{L}_{\psi^{-1}}) = 0$$

for all  $i > 0$ . By Lemma 4.1 (i), this implies that

$$H^i(Y \otimes_k \bar{k}, j_{Y!}(\mathcal{K}_{\chi^{-1}}[n]) \otimes g^* \mathcal{L}_{\psi^{-1}}) = 0$$

for all  $i > 0$ . By Poincaré duality, we have an isomorphism

$$H_c^i(Y \otimes_k \bar{k}, j_{Y!}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi) \cong \text{Hom} \left( H^{-i}(Y \otimes_k \bar{k}, j_{Y!}(\mathcal{K}_{\chi^{-1}}[n]) \otimes g^* \mathcal{L}_{\psi^{-1}}), \overline{\mathbf{Q}}_l(-n) \right).$$

Here we use the fact that the Verdier dual of  $j_{Y!}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi$  is  $j_{Y!}(\mathcal{K}_{\chi^{-1}}[n]) \otimes g^* \mathcal{L}_{\psi^{-1}}(n)$ . It follows that

$$H_c^i(Y \otimes_k \bar{k}, j_{Y!}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi) = 0$$

also for all  $i < 0$ . Moreover, the main theorem of [D] ([D] 3.3.1 and 6.2.3) implies that  $H_c^0(Y \otimes_k \bar{k}, j_{Y!}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi)$  is pure of weight  $n$ . For any mixed object  $K$  in the derived category  $D_c^b(s, \overline{\mathbf{Q}}_l)$ , where  $s = \text{Spec } k$ , define its Poincaré polynomial to be

$$P(K) = \sum_{w \in \mathbf{Z}} \sum_{i \in \mathbf{Z}} (-1)^i e_{iw} T^w,$$

where  $e_{iw}$  is the number of eigenvalues with weight  $w$  counted with multiplicities of the geometric Frobenius element  $F$  in  $\text{Gal}(\bar{k}/k)$  acting on  $H^i(K_{\bar{s}})$ . We often write  $P(K)$  as  $P(K_{\bar{s}})$  by abuse of notations. By the above discussion, we have

$$P(R\Gamma_c(Y \otimes_k \bar{k}, j_{Y!}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi)) = bT^n,$$

where  $b = \dim(H_c^0(Y \otimes_k \bar{k}, j_{Y!}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi))$ . As  $Y$  is the disjoint union of  $Y \cap O_\sigma$  ( $\sigma \in \Sigma$ ), and the cone  $\sigma = 0$  corresponds to  $Y \cap \mathbf{T}_A^n \cong \mathbf{T}_k^n$ , we have

$$P(R\Gamma_c(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi))$$

$$\begin{aligned}
&= (-1)^n P(R\Gamma_c(\mathbf{T}_{\bar{k}}^n, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi)) \\
&= (-1)^n \left( P(R\Gamma_c(Y \otimes_k \bar{k}, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi)) - \sum_{\sigma \neq 0} P(R\Gamma_c((Y \cap O_\sigma) \otimes_k \bar{k}, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi)) \right) \\
&= (-1)^n \left( bT^n - \sum_{\sigma \neq 0} P(R\Gamma_c((Y \cap O_\sigma) \otimes_k \bar{k}, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi)) \right).
\end{aligned}$$

We claim that if  $0 \notin F_{\Delta_\infty(f)}(\sigma)$ , then

$$H_c^i((Y \cap O_\sigma) \otimes_k \bar{k}, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi) = 0$$

for all  $i$ . Indeed, by Lemma 3.7,  $R^j(g|_{Y \cap O_\sigma})_!(j_{Y!*}(\mathcal{K}_\chi[n]) \otimes \mathcal{L}_\psi)$  are constant sheaves on  $\mathbf{A}_k^1$  for all  $j$ . It follows that

$$H_c^i(\mathbf{A}_k^1, R^j(g|_{Y \cap O_\sigma})_!(j_{Y!*}(\mathcal{K}_\chi[n]) \otimes \mathcal{L}_\psi)) = 0$$

for all  $i, j$ . Our claim then follows from the spectral sequence

$$\begin{aligned}
E_2^{ij} &= H_c^i(\mathbf{A}_k^1, R^j(g|_{Y \cap O_\sigma})_!(j_{Y!*}(\mathcal{K}_\chi[n]) \otimes \mathcal{L}_\psi)) \cong H_c^i(\mathbf{A}_k^1, R^j(g|_{Y \cap O_\sigma})_!(j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi)) \\
&\Rightarrow H_c^{i+j}((Y \cap O_\sigma) \otimes_k \bar{k}, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi).
\end{aligned}$$

So we have

$$P(R\Gamma_c(\mathbf{T}_{\bar{k}}^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)) = (-1)^n \left( bT^n - \sum_{\sigma \neq 0, 0 \in F_{\Delta_\infty(f)}(\sigma)} P(R\Gamma_c((Y \cap O_\sigma) \otimes_k \bar{k}, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi)) \right).$$

For those  $\sigma$  with  $0 \in F_{\Delta_\infty(f)}(\sigma)$ , by Lemma 3.2 (iii),  $Y \cap O_\sigma$  can be identified with  $O_{\sigma_k}$  and  $g|_{Y \cap O_\sigma} : Y \cap O_\sigma \rightarrow \mathbf{A}_k^1$  can be identified with  $f_{\tau_\sigma} : O_{\sigma_k} \rightarrow \mathbf{A}_k^1$ , where  $\tau_\sigma = F_{\Delta_\infty(f)}(\sigma)$ . Using Lemma 3.3, one can check  $(j_{Y!*}(\mathcal{K}_\chi[n]))|_{Y \cap O_\sigma}$  is identified with  $(j_{!*}(\mathcal{K}_\chi[n]))|_{O_{\sigma_k}}$ . So we have

$$P(R\Gamma_c(\mathbf{T}_{\bar{k}}^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)) = (-1)^n \left( bT^n - \sum_{\sigma \neq 0, 0 \in F_{\Delta_\infty(f)}(\sigma)} P(R\Gamma_c(O_{\sigma_k}, (j_{!*}(\mathcal{K}_\chi[n]))|_{O_{\sigma_k}} \otimes f_{\tau_\sigma}^* \mathcal{L}_\psi)) \right).$$

By Lemma 2.3, if  $\mathcal{K}_\chi$  is not the inverse image of a Kummer sheaf on  $O_{\sigma_k}$  under the projection

$$p_\sigma : \mathbf{T}_k^n = \text{Spec } k[\mathbf{Z}^n] \rightarrow O_{\sigma_k} = \text{Spec } k[\mathbf{Z}^n \cap \sigma^\perp],$$

then  $(j_{!*}(\mathcal{K}_\chi[n]))|_{O_{\sigma_k}}$  is acyclic. Let  $S$  be the set of those cones  $\sigma$  in  $\Sigma$  so that  $\sigma \neq 0, 0 \in F_{\Delta_\infty(f)}(\sigma)$ , and  $\mathcal{K}_\chi \cong p_\sigma^* \mathcal{K}_{\chi_\sigma}$  for a Kummer sheaf  $\mathcal{K}_{\chi_\sigma}$  on  $O_{\sigma_k}$ . For each  $\sigma \in S$ , let  $\delta_\sigma$  be the image of  $\tilde{\sigma}$  under the projection  $\mathbf{R}^n \rightarrow \mathbf{R}^n / \sigma^\perp$ , let  $j'_\sigma : \text{Spec } k[\mathbf{Z}^n / \mathbf{Z}^n \cap \sigma^\perp] \rightarrow X_k(\Sigma(\delta_\sigma))$  be the immersion

of the open dense torus in  $X_k(\Sigma(\delta_\sigma))$ , let  $x_\sigma$  be the distinguished point in  $X_k(\Sigma(\delta_\sigma))$ , and let  $\pi_\sigma : O_{\sigma k} \rightarrow \text{Spec } k$  be the structure morphism. Then by Lemma 2.3, we have

$$(j'_! (\mathcal{K}_\chi[n]))|_{O_{\sigma k}} \cong (\mathcal{K}_{\chi_\sigma}[n - \dim(\sigma)]) \otimes \pi_\sigma^* x_\sigma^* (j'_{\sigma! *} (\overline{\mathbf{Q}}_l[\dim(\sigma)])).$$

Therefore we have

$$\begin{aligned} & P(R\Gamma_c(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)) \\ &= (-1)^n \left( bT^n - \sum_{\sigma \in S} P(R\Gamma_c(O_{\sigma \bar{k}}, (\mathcal{K}_{\chi_\sigma}[n - \dim(\sigma)]) \otimes \pi_\sigma^* x_\sigma^* (j'_{\sigma! *} (\overline{\mathbf{Q}}_l[\dim(\sigma)])) \otimes f_{\tau_\sigma}^* \mathcal{L}_\psi) \right) \\ &= (-1)^n \left( bT^n - \sum_{\sigma \in S} (-1)^{n - \dim(\sigma)} P(R\Gamma_c(O_{\sigma \bar{k}}, \mathcal{K}_{\chi_\sigma} \otimes f_{\tau_\sigma}^* \mathcal{L}_\psi)) P(x_\sigma^* (j'_{\sigma! *} (\overline{\mathbf{Q}}_l[\dim(\sigma)]))) \right) \end{aligned}$$

By Remark 2.2 and the fact that the polynomial  $\alpha(\delta_\sigma)$  involves only even powers of  $T$ , we have

$$P(x_\sigma^* (j'_{\sigma! *} (\overline{\mathbf{Q}}_l[\dim(\sigma)]))) = (-1)^{\dim(\sigma)} \alpha(\delta_\sigma).$$

So we get

$$P(R\Gamma_c(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)) = (-1)^n \left( bT^n - \sum_{\sigma \in S} (-1)^n P(R\Gamma_c(O_{\sigma \bar{k}}, \mathcal{K}_{\chi_\sigma} \otimes f_{\tau_\sigma}^* \mathcal{L}_\psi)) \alpha(\delta_\sigma) \right).$$

By (i), we have

$$\begin{aligned} E(\mathbf{T}_k^n, f, \chi) &= (-1)^n P(R\Gamma_c(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)) \\ E(O_{\sigma k}, f_{\tau_\sigma}, \chi_\sigma) &= (-1)^{n - \dim(\sigma)} P(R\Gamma_c(O_{\sigma \bar{k}}, \mathcal{K}_{\chi_\sigma} \otimes f_{\tau_\sigma}^* \mathcal{L}_\psi)). \end{aligned}$$

So we finally get

$$E(\mathbf{T}_k^n, f, \chi) = bT^n - \sum_{\sigma \in S} (-1)^{\dim(\sigma)} E(O_{\sigma k}, f_{\tau_\sigma}, \chi_\sigma) \alpha(\delta_\sigma).$$

To get an explicit formula for  $E(\mathbf{T}_k^n, f, \chi)$ , we now take  $\Sigma = \Sigma(\Delta_\infty(f))$ . By Proposition 1.2,

$$\tau \mapsto (\text{cone}_{\Delta_\infty(f)}(\tau))^\vee$$

defines a one-to-one correspondence between faces of  $\Delta_\infty(f)$  and cones in  $\Sigma(\Delta_\infty(f))$ . Moreover,

for  $\sigma_\tau = (\text{cone}_{\Delta_\infty(f)}(\tau))^\vee$ , we have

$$\begin{aligned} \tau &= F_{\Delta_\infty(f)}(\sigma_\tau), \\ \dim(\sigma_\tau) &= n - \dim(\tau), \\ \sigma_\tau^\perp &= \check{\sigma}_\tau \cap (-\check{\sigma}_\tau) = (\text{cone}_{\Delta_\infty(f)}(\tau)) \cap (-\text{cone}_{\Delta_\infty(f)}(\tau)) = \text{span}(\tau - \tau), \end{aligned}$$

and the image of  $\check{\sigma}_\tau = \text{cone}_{\Delta_\infty(f)}(\tau)$  in  $\mathbf{R}^n/\sigma_\tau^\perp$  is just  $\text{cone}_{\Delta_\infty(f)}^\circ(\tau)$ . Let  $T$  be the set of faces  $\tau$  of  $\Delta_\infty(f)$  so that  $\tau \neq \Delta_\infty(f)$ ,  $0 \in \tau$ , and  $\mathcal{K}_\chi \cong p_\tau^* \mathcal{K}_\tau$  for a Kummer sheaf  $\mathcal{K}_\tau$  on  $\mathbf{T}_\tau = \text{Spec } k[\mathbf{Z}^n \cap \text{span}(\tau - \tau)]$ , where

$$p_\tau : \mathbf{T}_k^n = \text{Spec } k[\mathbf{Z}^n] \rightarrow \mathbf{T}_\tau = \text{Spec } k[\mathbf{Z}^n \cap \text{span}(\tau - \tau)]$$

is the projection. Note that  $T$  corresponds to the set  $S$  under the one-to-one correspondence  $\tau \mapsto (\text{cone}_{\Delta_\infty(f)}(\tau))^\vee$ . We then have

$$E(\mathbf{T}_k^n, f, \chi) = bT^n - \sum_{\tau \in T} (-1)^{n-\dim(\tau)} E(\mathbf{T}_\tau, f_\tau, \chi_\tau) \alpha(\text{cone}_{\Delta_\infty(f)}^\circ(\tau)).$$

By (ii), we have

$$\begin{aligned} E(\mathbf{T}_k^n, f, \chi)(1) &= n! \text{vol}(\Delta_\infty(f)), \\ E(\mathbf{T}_\tau, f_\tau, \chi_\tau)(1) &= (\dim(\tau))! \text{vol}(\tau). \end{aligned}$$

Evaluating the above equality at 1, we get

$$b = n! \text{vol}(\Delta_\infty(f)) + \sum_{\tau \in T} (-1)^{n-\dim(\tau)} (\dim(\tau))! \text{vol}(\tau) \alpha(\text{cone}_{\Delta_\infty(f)}^\circ(\tau))(1),$$

that is,  $b = e(\Delta_\infty(f), \chi)$ . So we have

$$E(\mathbf{T}_k^n, f, \chi) = e(\Delta_\infty(f), \chi)T^n - \sum_{\tau \in T} (-1)^{n-\dim(\tau)} E(\mathbf{T}_\tau, f_\tau, \chi_\tau) \alpha(\text{cone}_{\Delta_\infty(f)}^\circ(\tau)).$$

Using this expression, the definition of  $E(\Delta_\infty(f), \chi)$ , and induction on  $\dim(\Delta_\infty(f))$ , we get

$$E(\mathbf{T}_k^n, f, \chi) = E(\Delta_\infty(f), \chi).$$

By [D] 3.3.1 and 3.3.3,  $H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$  is mixed with weights between 0 and  $n$ . So  $E(\mathbf{T}_k^n, f, \chi)$  is a polynomial of degree  $\leq n$ . By the definition of  $\alpha$ , we have

$$\deg(\alpha(\text{cone}_{\Delta_\infty(f)}^\circ(\tau))) \leq \dim(\text{cone}_{\Delta_\infty(f)}^\circ(\tau)) - 1 = n - \dim(\tau) - 1.$$

Moreover, we have

$$\deg(E(\mathbf{T}_\tau, f_\tau, \chi_\tau)) \leq \dim(\tau).$$

It now follows from the expression

$$E(\mathbf{T}_k^n, f, \chi) = e(\Delta_\infty(f), \chi)T^n - \sum_{\tau \in T} (-1)^{n-\dim(\tau)} E(\mathbf{T}_\tau, f_\tau, \chi_\tau) \alpha(\text{cone}_{\Delta_\infty(f)}^\circ(\tau))$$



that

$$e_n = e(\Delta_\infty(f), \chi).$$

(iv) Since 0 is an interior point of  $\Delta_\infty(f)$ , for any nonzero  $\sigma \in \Sigma$ , we have  $0 \notin F_{\Delta_\infty(f)}(\sigma)$ . We have seen in the proof of (iii) that this implies

$$H_c^i((Y \cap O_\sigma) \otimes_k \bar{k}, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi) = 0$$

for all  $i$ . Since  $Y - Y \cap \mathbf{T}_A^n$  is the disjoint union of  $Y \cap O_\sigma$  for nonzero  $\sigma$ , we have

$$H_c^i((Y - Y \cap \mathbf{T}_A^n) \otimes_k \bar{k}, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi) = 0$$

for all  $i$ . (We can also apply Lemma 3.5 (iii) to  $Y - Y \cap \mathbf{T}_A^n$ ,  $J = \{\sigma \in \Sigma \mid \dim(\sigma) = 1\}$ , and the closed subschemes  $(Y - Y \cap \mathbf{T}_A^n) \cap V(\sigma) = Y \cap V(\sigma)$  ( $\sigma \in J$ ).) So we have

$$H_c^i((Y \cap \mathbf{T}_A^n) \otimes_k \bar{k}, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi) \cong H_c^i(Y \otimes_k \bar{k}, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi)$$

for all  $i$ , that is,

$$H_c^{i+n}(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi) \cong H_c^i(Y \otimes_k \bar{k}, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi)$$

for  $i$ . In the proof of (iii), we have seen  $H_c^0(Y \otimes_k \bar{k}, j_{Y!*}(\mathcal{K}_\chi[n]) \otimes g^* \mathcal{L}_\psi)$  is pure of weight  $n$ . So  $H_c^n(\mathbf{T}_k^n, \mathcal{K}_\chi \otimes f^* \mathcal{L}_\psi)$  is pure of weight  $n$ .

## References.

- [AS] A. Adolphson and S. Sperber, *Twisted exponential sums and Newton polyhedra*, J. Reine Angew. Math. 443 (1993), 151-177.
- [BBD] A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, in *Analyse et Topologie sur les Espace Singuliers (I)*, Astérisque 100 (1980).
- [D] P. Deligne, *La conjecture de Weil II*, Publ. Math. IHES 52 (1980), 137-252.
- [DL] J. Denef and F. Loeser, *Weights of exponential sums, intersection cohomology, and Newton polyhedra*, Invent. Math. 106 (1991), 275-294.

- [F] W. Fulton, *Introduction to toric varieties*, Annals of Math. Studies (131), Princeton University Press 1993.
- [I1] L. Illusie, *Théorie de Brauer et Caractéristique d'Euler-Poincaré*, in *Caractéristique d'Euler-Poincaré*, Astérisque 82-83 (1981), 161-172.
- [I2] L. Illusie, *Autour du théorème de monodromie locale*, in *Périodes  $p$ -adiques*, Astérisque 223 (1994), 9-57.
- [SGA] Séminaire de Géométrie Algébrique du Bois-Marie.
- [S] R. Stanley, *Generalized  $H$ -vectors, intersection cohomology of toric varieties, and related results*, in *Commutative Algebra and Combinatorics* edited by N. Nagata and H. Matsumura, Adv. Stud. Pure Math., vol 11, 187-213, Amsterdam, New York, North-Holland 1987.
- [SGA 1] Revêtements étales et groupe fondamental, by Grothendieck, *Lecture Notes in Mathematics* 224, Springer-Verlag (1971).
- [SGA 4] Théorie des topos et cohomologie étale des schémas, by M. Artin, A. Grothendieck and J.-L. Verdier, *Lecture Notes in Mathematics* 269, 270, 305, Springer-Verlag (1972-1973).
- [SGA 4 $\frac{1}{2}$ ] Cohomologie étale, by P. Deligne, *Lecture Notes in Mathematics* 569, Springer-Verlag (1977).
- [SGA 5] Cohomologie  $l$ -adique et fonctions L, by Grothendieck, *Lecture Notes in Mathematics* 589, Springer-Verlag (1977).
- [SGA 7] Groupes de monodromie en géométrie algébrique, I by A. Grothendieck, II by P. Deligne and N. Katz, *Lecture Notes in Mathematics* 288, 340, Springer-Verlag (1972-1973).